Variational data assimilation problems for general ocean circulation models and numerical methods for these problems

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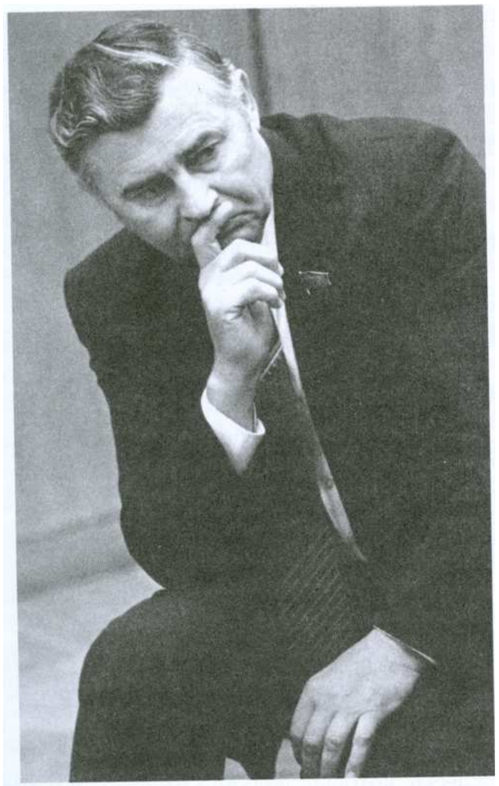
Introduction



In INM RAS:

- **1991-2003** – Development of the general methodology
- **2004-2005** – First works on assimilation of sea surface temperature.
- **2005-2006** – Study of problems for semidiscrete models
- **2007** – Existence theorems for "continuous equations"
- **2007** – Methods and technology for solving inverse hydrothermodynamics problem in ocean with variational assimilation of sea surface temperature.
- **2005-2010** – Study of the inverse and variational data assimilation problems ocean dynamics theory for semidiscrete models.

Variational data assimilation problems for general ocean circulation models

- Solvability results for forward problems
- Solvability of inverse and variational data assimilation problems for general models
- Development of the general methodology for solving the problems

- Splitting methods and semidiscrete models
- Solvability of inverse and variational data assimilation problems for semidiscrete models
- Methodology for solving the problems using the adjoint equation technique

- Numerical methods for interpolation of the data to be assimilated
- Preparation of sea surface temperature data and data from the ARGO system
- Development of the Data base "World Ocean – INM RAS"

- Development of the Data assimilation system for PC
- Development of the Data assimilation system for system of PC based on Internet technology

Theoretical background of the study

- Mathematical models and complex system theory
- Optimal control theory
- Operator theory and boundary value problems theory
- Iterative algorithms theory
- Adjoint equation theory
- Modern numerical algorithms

1. Mathematical model of the ocean hydrothermodynamics

$$\frac{d\vec{u}}{dt} + \left[\begin{array}{cc} 0 & -f \\ f & 0 \end{array} \right] \vec{u} - g \cdot grad\xi + A_u\vec{u} + (A_k)^2\vec{u} = \vec{f} - \frac{1}{\rho_0} gradP_a - \frac{g}{\rho_0} grad \int\limits_0^z \rho_1(T,S)dz',$$

$$\frac{\partial\xi}{\partial t} - m\frac{\partial}{\partial x}(\int\limits_0^H \Theta(z)udz) - m\frac{\partial}{\partial y}(\int\limits_0^H \Theta(z)\frac{n}{m}vdz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T, \ \frac{dS}{dt} + A_S S = f_S,$$

where $\vec{u} = (u,v)$ and $\bar{f} = g \cdot gradG$, $\Theta(z) \equiv \frac{r(z)}{R}$, $\quad r = R - z, \quad 0 < z < H, x \equiv \lambda, y \equiv \theta, n \equiv 1/r, \ m \equiv 1/(r\cos\theta)$. The functions G, f_3, ξ_0 $\equiv \xi$ at $t = 0$ will be "additional unknowns which must be calculated too.

Boundary conditions on the "sea surface" $\Gamma_S \equiv \Omega$ at $z = 0$:

$$
\begin{cases}
\left(\int\limits_0^H \Theta \vec{u} dz\right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\
U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \Big/ \rho_0, \; U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \Big/ \rho_0, \\
A_k u = 0, \quad A_k v = 0, \\
U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\
U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S,
\end{cases}
$$

where
$U_n = \vec{U} \cdot \vec{N}, \vec{U} = (u, v, w) \equiv (\vec{u}, w), \vec{N} = (n_1, n_2, n_3) \equiv (\vec{n}, n_3), U_n^{(-)} = (|U_n| - U_n)/2.$
The boundary function d_T, d_S or Q_T, Q_S can be unknown also.

With the function $\phi = (u, v, \xi, T, S)$ known, we calculate

$$w(x,y,z,t) = \frac{1}{r}\left(m\frac{\partial}{\partial x}\left(\int_z^H r u dz'\right) + m\frac{\partial}{\partial y}\left(\frac{n}{m}\int_z^H r v dz'\right)\right), (x,y,t) \in \Omega \times (0, \bar{t}),$$

$$P(x,y,z,t) = P_a(x,y,t) + \rho_0 g(z-\xi) + \int_0^z g\rho_1(T,S)dz'.$$

Note, that for $U_n \equiv \underline{U} \cdot \underline{N}$ (here $U = (u,v,w)$) we always have

$$U_n = 0 \text{ on } \Gamma_{c,w} \cup \Gamma_H .$$

2. Approximation by splitting method

- General theory of splitting methods: G.I. Marchuk , N.N. Yanenko, A.A. Samarsky.
- Splitting method in data assimilation: Marchuk G.I., Zalesny V.B. (1993), M. Wenzel, V.B. Zalesny (1996), V.B. Zalesny (2005).
- Studies of inverse and assimilation problems for semidiscrete models in the ocean dynamics: Agoshkov V.I. (2005-2008).
- Studies of class of inverse and data assimilation problems for ocean dynamics models obtained by splitting method: Agoshkov V.I. (2005, 2006), Zalesny V.B. (2008, 2010), Agoshkov V.I., Parmuzin E.I., Zakharova N.B. (2010).

Problem I

Step 1. We consider the system:

$$\begin{cases} T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\ T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\ T_j \equiv T \text{ on } D \times (t_{j-1}, t_j), \end{cases}$$

where $\Gamma_w = \Gamma_{w,c} \cup \Gamma_{w,op}$ - the "vertical lateral boundary", Γ_H - "the ocean bottom".

We consider the subproblem for T in the operator form as

$$(T)_t + LT = \mathcal{F} + BQ, \quad t \in (t_{j-1}, t_j),$$

$$T = T_{j-1}, \quad j = 1, 2, \ldots, J,$$

and introduce the additional approximation by the splitting methods:

Step 1.1:

$$(T_1)_t + L_1 T_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j),$$

$$T_1 = T_{j-1} \quad \text{at} \quad t = t_{j-1}$$

Step 1.2:

$$(T_2)_t + L_2 T_2 = \mathcal{F}_2 + BQ, \quad t \in (t_{j-1}, t_j),$$

$$T_2(t_{j-1}) = T_1(t_j).$$

$$T_2(t_j) \equiv T_j \cong T \quad \text{at} \ \ t = t_j.$$

The classical form of the subproblem for $T_2 \equiv T$ is given by:

$$\begin{cases} T_t + \frac{1}{2}\left(w_1 \frac{\partial T}{\partial z} + \frac{1}{r^2}\frac{\partial (r^2 w_1 T)}{\partial z}\right) - \frac{1}{r^2}\frac{\partial}{\partial z} r^2 \nu_T \frac{\partial T}{\partial z} = f_T \ \text{ in } D \ \text{ at } \ t \in (t_{j-1}, t_j), \\ T = T_1(t_j) \quad \text{at} \ \ t = t_{j-1}, \\ -\nu_T \frac{\partial T}{\partial z} = Q \ \text{ at } \ z = 0, \\ \nu_T \frac{\partial T}{\partial z} = 0 \ \text{ at } \ z = H, \end{cases}$$

where

$$\bar{U}_n^{(-)} = \frac{|\bar{U}_n| - \bar{U}_n}{2} = \frac{1}{2}(|\bar{w}_1| + \bar{w}_1) = \frac{1}{2}(|\bar{w}| + \bar{w}) \ \text{ at } \ z = 0,$$

$$Q \equiv Q_T - \gamma_T (T - T_a) - \bar{U}_n^{(-)} T + \bar{U}_n^{(-)} d_T.$$

Step 2.

$$
\begin{cases}
S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j),\\
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D,\\
\bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j),\\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j),\\
\bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j),\\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j),\\
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

We rewrite the subproblem for S in the operator form as

$$(S)_t + LS = \mathcal{F} + BQ, \quad t \in (t_{j-1}, t_j),$$

$$S = S_{j-1}, \quad j = 1, 2, \dots, J,$$

and introduce the additional approximation by the splitting methods:

Step 1.1:

$$(S_1)_t + L_1 S_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j),$$

$$S_1 = S_{j-1} \quad \text{at} \quad t = t_{j-1}$$

Step 1.2:

$$(S_2)_t + L_2 S_2 = \mathcal{F}_2 + BQ, \quad t \in (t_{j-1}, t_j),$$

$$S_2(t_{j-1}) = S_1(t_j).$$

$$S_2(t_j) \equiv S_j \cong S \quad \text{at} \ \ t = t_j.$$

The classical form of the subproblem for $S_2 \equiv S$ is given by:

$$\begin{cases} S_t + \dfrac{1}{2}\left(w_1 \dfrac{\partial S}{\partial z} + \dfrac{1}{r^2}\dfrac{\partial (r^2 w_1 S)}{\partial z}\right) - \dfrac{1}{r^2}\dfrac{\partial}{\partial z} r^2 \nu_S \dfrac{\partial S}{\partial z} = f_S \ \text{ in } D \ \text{ at } \ t \in (t_{j-1}, t_j), \\ S = S_1(t_j) \quad \text{at} \ \ t = t_{j-1}, \\ -\nu_S \dfrac{\partial S}{\partial z} = Q \ \text{ at } \ z = 0, \\ \nu_S \dfrac{\partial S}{\partial z} = 0 \ \text{ at } \ z = H, \end{cases}$$

where

$$\bar{U}_n^{(-)} = \frac{|\bar{U}_n| - \bar{U}_n}{2} = \frac{1}{2}(|\bar{w}_1| + \bar{w}_1) = \frac{1}{2}(|\bar{w}| + \bar{w}) \ \text{ at } \ z = 0,$$

$$Q \equiv Q_S - \gamma_S (S - S_a) - \bar{U}_n^{(-)} S + \bar{U}_n^{(-)} d_S .$$

Step 3: The subproblems for the velocity components

Step 3.1

$$\begin{cases} \underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad}G - \frac{1}{\rho_0}\mathbf{grad}\left(P_a + g \int\limits_0^z \rho_1(\bar{T}, \bar{S}) dz' \right) \\ \text{ in } D \times (t_{j-1}, t_j), \\ \xi_t - \mathbf{div}\left(\int\limits_0^H \Theta \underline{u}^{(1)} dz \right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\ \underline{u}^{(1)} = \underline{u}_{j-1}, \ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\ \left(\int\limits_0^H \Theta \underline{u}^{(1)} dz \right) \cdot n + \beta_0 m_{op}\sqrt{gH}\xi = m_{op}\sqrt{gH} d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\ \underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D \end{cases}$$

If we write down $\underline{u}^{(1)}$ in the following form: $\underline{u}^{(1)} = \underline{U}^{(1)}(\lambda, \theta, t) + \underline{u}'(\lambda, \theta, z, t)$ where

$$\underline{U}^{(1)} = \frac{1}{H_1} \int_0^H \Theta \underline{u}^{(1)} dz, \qquad H_1 = \int_0^H \Theta dz,$$

then Step 3.1 is reduced to two subproblems for the functions $\underline{U}^{(1)}, \underline{u}_1'$.

Step 3.1

First of them is "The ocean tide theory problem":

$$\begin{cases} \underline{U}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{U}^{(1)} - g \,\mathrm{grad}\, \xi = g \,\mathrm{grad}\, G - \underline{I} \quad \text{in} \;\; D \times (t_{j-1}, t_j) \\ \xi_t - \mathrm{div}(H_1 \underline{U}^{(1)}) = f_3 \quad \text{in} \;\; \Omega \times (t_{j-1}, t_j) \\ \underline{U}^{(1)}(t_{j-1}) = \frac{1}{H_1} \int_0^H \Theta \underline{u}_{j-1} dz, \qquad \xi(t_{j-1}) = \xi_{j-1} \quad \text{in } \Omega \\ \\ (H_1 \underline{U}^{(1)}) \cdot \underline{n} + \beta_0 m_{\mathrm{op}} \sqrt{gH} \xi = m_{\mathrm{op}} \sqrt{gH} d_s \end{cases}$$

where

$$\underline{I} = (I_\lambda, I_\theta) = \frac{1}{\rho_0} \Big(\mathrm{grad}\, P_a + g \frac{1}{H_1} \int_0^H \Theta dz \int_0^z \mathrm{grad}\, \rho_1(\bar{T}, \bar{S}) dz' \Big).$$

The study and solution of this subproblem and its adjoint problem have the crusial meaning for one of the inverse and data assimilation problems studied.

The second subproblem is :

$$
\begin{cases}
(\underline{u}_1')_t + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}_1' = \frac{g}{\rho_0}\Big(\frac{1}{H_1} \int_0^H \Theta dz \int_0^z \operatorname{grad} \rho_1(\bar{T}, \bar{S}) dz' \\
- \displaystyle\int_0^z \operatorname{grad} \rho_1(\bar{T}, \bar{S}) dz' \Big) \\
\underline{u}_1'(t_{j-1}) = \underline{u}_{j-1} - \dfrac{1}{H_1} \displaystyle\int_0^H \Theta u_{j-1} dz
\end{cases}
$$

Step 3.2

$$\begin{cases} \underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ при } t = t_{j-1} \text{ in } D, \\ \underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D, \end{cases}$$

Step 3.3

$$\begin{cases} \underline{u}_t^{(3)} + (\bar{U}, \mathbf{Grad})\underline{u}^{(3)} - \mathbf{Div}(\hat{a}_u \cdot \mathbf{Grad}\ )\underline{u}^{(3)} + (A_k)^2\underline{u}^{(3)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(3)} = \underline{u}^{(2)} \text{ at } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)}\underline{u}^{(3)} - \nu_u \frac{\partial \underline{u}^{(3)}}{\partial z} - k_{33}\frac{\partial}{\partial z}(A_k\underline{u}^{(3)}) = \frac{\underline{\tau}^{(a)}}{\rho_0}, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ U_n^{(3)} = 0, \frac{\partial U^{(3)}}{\partial N_u} \cdot \bar{\underline{\tau}}_w + \left(\frac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \underline{\tau}_w = 0, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)}(\tilde{U}^{(3)} \cdot \underline{N}) + \frac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{N} + \left(\frac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \bar{N} = \bar{U}_n^{(-)} d, A_k\underline{u}^{(3)} = 0 \text{on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)}(\tilde{U}^{(3)} \cdot \bar{\tau}_w) + \frac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{\tau}_w + \left(\frac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \underline{\tau}_w = 0, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \frac{\partial \underline{u}^{(3)}}{\partial N_u} = \frac{\tau^{(b)}}{\rho_0} \text{ on } \Gamma_H \times (t_{j-1}, t_j), \end{cases}$$

where

$$\underline{u}^{(3)} = (u^{(3)}, v^{(3)}),\ \ \tau^{(a)} = (\tau_x^{(a)}, \tau_y^{(a)}),$$
$$U^{(3)} = (u^{(3)}, w^{(3)}(u^{(3)}, v^{(3)})),\ \ \tilde{U}^{(3)} = (u^{(3)}, 0),\ \ \tau^{(b)} = (\tau_x^{(b)}, \tau_y^{(b)}).$$

3. Inverse and variational data assimilation problems, I

Let us assume, that the unique function which is obtained by observation data processing is the function ξ_{obs} on $\bar{\Omega} \equiv \Omega \cup \partial\Omega$ at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. Let by physical meaning this function is an approximation to sea level function ξ on Ω, i.e on the boundary, when $z = 0$. We permit that the function ξ_{obs} is known only on the part of $\Omega \times (0, \bar{t})$ and we define a support of this function as m_0. Beyond of this area we suppose function ξ_{obs} is trivial.

Let the functions G, f_3, ξ_0 are "additional unknown functions" and we state the following inverse problem - **Problem Inv 1**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* G, f_3, ξ_0, *such that,* $m_0(\xi - \xi_{obs}) = 0$.

To study this inverse problem we apply general methodology for solving data assimilation problems (Agoshkov V., 2003) and classical results of the inverse problem theory (A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, V.V. Vasin, V.G. Romanov, M.V. Klibanov,Yu.E. Anikonov,S.I. Kabanikhin, A.Hasanov, V.G. Yakhno).

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(\xi_0, G, f_3, \Phi) = \frac{1}{2}\Big\{\alpha_0 \bar{t}\left\|\xi_0 - \xi^{(0)}\right\|^2_{L_2(g;\Omega)} + \alpha_f \left\|f_3 - f_3^{(0)}\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}$$

$$+\alpha_G \left\|G - G^{(0)}\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}\Big\} + \Im_0(\Phi) = \sum_{j=1}^{J}\int_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(\xi) = \frac{1}{2}\left\|m_0(\xi - \xi_{\text{obs}})\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\Big\{\alpha_0 \Delta t_j \left\|\xi_0 - \xi^{(0)}\right\|^2_{L_2(g;\Omega)} + \alpha_f \left\|f_3 - f_3^{(0)}\right\|^2_{L_2(g;\Omega)}(t)$$

$$+\alpha_G \left\|G - G^{(0)}\right\|^2_{L_2(g;\Omega)}(t) + \left\|m_0(\xi - \xi_{\text{obs}})\right\|^2_{L_2(g;\Omega)}(t)\Big\}.$$

Here $\alpha \equiv (\alpha_0, \alpha_f, \alpha_G)$, $\alpha_0 \geq 0$, $\alpha_f \geq 0$, $\alpha_G \geq 0$ are regularization parameters that may be dimensional values. Furthermore, it is possible to specify α_f, α_G depending on $\alpha_0 \geq 0$, (for instance, $\alpha_G = \alpha_0$, $\alpha_f = \alpha_0 \bar{t}^2$, etc.).

We can formulate the data assimilation problem - **Problem A 1**: *find the solution* ϕ *of the Problem* I *and function* G, f_3, ξ_0, *such that, the cost functional is minimal on the set of the solutions.*

Let us consider the problem on the first time step (t_0, t_1). Then the optimality conditions are:

$$\begin{cases} t_1\alpha_0(\xi_0 - \xi^{(0)}) + \xi^*(t_0) = 0 \quad \text{in } \Omega \\ \alpha_f(f_3 - f_3^{(0)}) + \xi^* = 0 \quad \text{in } \Omega \times (t_0, t_1) \\ \alpha_G(G - G^{(0)}) - \operatorname{div}\Big(\int_0^{\mathrm{H}} \Theta \underline{u}_1^* dz\Big) = 0 \quad \text{in } \Omega \times (t_0, t_1), \end{cases}$$

where $\xi^*, \underline{u}_1^*$ are the solution of the adjoint problem:

$$\begin{cases} -(\underline{u}_1^*)_t - \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}_1^* + g \,\mathrm{grad}\, \xi^* = 0 \quad \text{in } D \times (t_0, t_1) \\ -\xi_t^* + \operatorname{div}\Big(\int_0^H \Theta \underline{u}_1^* dz\Big) = m_0(\xi - \xi_{\mathrm{obs}}) \quad \text{in } \Omega \times (t_0, t_1) \\ -\Big(\int_0^H \Theta \underline{u}_1^* dz\Big) \cdot \underline{n} + \beta_0 m_{\mathrm{op}} \sqrt{gH} \xi^* = 0 \quad \text{on } \partial\Omega \times (t_0, t_1) \\ \xi^* = 0, \qquad \underline{u}_1^* = 0 \quad \text{at } t = t_1 \end{cases}$$

(or here $\xi^* = m_0(\xi - \xi_{\mathrm{obs}})(t_1)$ at $t = t_1$).

Definition : Problem Inv 1 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\phi) < \epsilon$.

Proposition 1. If $\text{supp}(\xi_{\text{obs}}) = \bar{\Omega} \times [t_0, t_1]$ and $(G, f_3)_{L_2(g;\Omega)} = 0 \; \forall t$ then Problem Inv1 is uniquely and densely solvable. The solution of Problem A1 can be taken as an approximate solution of Problem Inv1 for sufficiently small α.

Proposition 2. If $\text{mes}(\partial\Omega \cap \Gamma_{w,\text{op}}) > 0$ and the function G is sought additionally only then Problem Inv 1 is densely solvable.

Proposition 3. If $\text{mes}(\text{supp}(\xi_{\text{obs}})) > 0$ and the function f_3 is sought additionally only then Problem Inv 1 is densely solvable.

Iterative process

For numerical implementation of the algorithm of solving the whole problem in (t_0, t_1) it is sufficient to solve two initial-boundary problems for parabolic equations (after that T and S will be defined in $D \times (t_0, t_1)$) and carry out Step 3 including the data assimilation block. A numerical solution of the problem at Step 3 can be obtained by the following iterative algorithm: if $f_3^{(k)}, G^{(k)}, \xi_0^{(k)}$ are defined, we solve the subproblems from the Step 3 for $\xi_0 = \xi_0^{(k)}$, $f_3 = f_3^{(k)}$, $G = G^{(k)}$ and then solve adjoint problem and compute the new approximation $f_3^{(k+1)}, G^{(k+1)}, \xi_0^{(k+1)}$.

$$\begin{cases} \xi_0^{(k+1)} = \xi_0^{(k)} - \gamma_k(\alpha_0(\xi_0^{(k)} - \xi^{(0)}) + \xi^*(t_0)) \quad \text{in } \Omega \\ f_3^{(k+1)} = f_3^{(k)} - \gamma_k(\alpha_f(f_3^{(k)} - f_3^{(0)}) + \xi^*) \quad \text{in } \Omega \times (t_0, t_1) \\ G^{(k+1)} = G^{(k)} - \gamma_k\Big(\alpha_G(G^{(k)} - G^{(0)}) - \operatorname{div}\Big(\int_0^H \Theta u_1^* dz\Big)\Big) \\ \qquad \text{in } \Omega \times (t_0, t_1) \\ \text{provided that } \displaystyle\int_\Omega \mathrm{G}^{(i)} d\Omega = 0 \quad \forall i. \end{cases}$$

In the case of an appropriate selection of parameters $\{\gamma_k\}$ the iterative process converges. In virtue of the compact solvability property, the following values can be assumed an efficient choice of γ_k:

$$\gamma_k = \frac{\dfrac{1}{2}\displaystyle\int_{t_0}^{t_1}\int_{\Omega} m_0(\xi^{(k)} - \xi_{\mathrm{obs}})^2 d\Omega dt}{\left(\displaystyle\int_{\Omega}(\xi^*(t_0))^2 d\Omega + \int_{t_0}^{t_1}\int_{\Omega}(\xi^*)^2 d\Omega dt + \int_{t_0}^{t_1}\int_{\Omega}\left(\mathrm{div}\int_0^H \underline{u}_1^{*(k)}\Theta dz\right)^2 d\Omega dt\right)}.$$

After the criterion of stopping the iteration process is satisfied, it is necessary to used the computed $f_3^{(k+1)}, G^{(k+1)}, \xi_0^{(k+1)}$ to solve other subproblems from the Step 3 and to obtain an approximate solution to the whole problem in $D \times (t_0, t_1)$.
After solving all problems and implementing the iteration process in (t_0, t_1), the variation assimilation problem is solved similarly in the subsequent intervals $(t_{j-1}, t_j), j = 2, 3, \ldots$. **In view of the established properties of unique and dense solvability of the considered Problem Inv and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t}\,)$.**

4. Inverse and variational data assimilation problems, II

Problem Inv 2

Assume that the sea surface temperature (SST), observed on a subset $\Omega^{(j)}$ of Ω, is denoted by $T_{obs} \equiv T_{obs}^{(j)}$ when $t \in (t_{j-1}, t_j), m_0^{(j)}$ is the characteristic function of this subset $(j = 1, 2, ..., J)$. Considering the boundary condition for T at $z = 0$ we write it in the following form:

$$-\nu_T \frac{\partial T}{\partial z} = Q \; at \; z = 0 \; on \, \Omega^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} T - \nu_T \frac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \; at \;\; z = 0 \; on \, (\Omega \backslash \Omega^{(j)}) \times (t_{j-1}, t_j),$$

where the function $Q \equiv Q^{(j)} (j = 1, 2, ..., J)$.

Let the functions $Q^{(j)}$ are "additional unknown functions" and we state the following inverse problem - **Problem Inv 2**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* $Q^{(j)}$, *such that,* $m_0^{(j)} (T - T_{obs}^{(j)}) = 0$ on $\Omega, j = 1, 2, ..., J$.

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(Q,\Phi) = \frac{1}{2}\int_0^{\bar t}\int_\Omega \alpha|Q - Q^{(0)}|^2 d\Omega dt + \Im_0(\Phi) = \sum_{j=1}^{J}\int_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(Q) = \frac{1}{2}\int_0^{\bar t}\int_{\Omega_0(t)} m_0|T - T_{obs}|^2 d\Omega dt,$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{\Omega^{(j)}} \alpha|Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{\Omega^{(j)}} m_0^{(j)}|T - T_{obs}^{(j)}|^2 d\Omega dt.$$

Here $\alpha \geq 0$, is a "regularization"or "penalty"function ,that may be constant.

Data assimilation problem - **Problem A 2**: *find the solution* ϕ *of the Problem* I *and functions* $\{Q^{(j)}\}$, *such that, the cost functional is minimal on the set of the solutions.*

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. The method can be discribed as follows:

STEP 1. We solve system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations. This system consists of equations for T_1, T_2, Q and system of adjoint equations:

$$\begin{cases} -(T_2^*)_t + L_2^* T_2^* = B^* m_0^{(1)} (T - T_{obs}^{(1)}) \quad \text{in} \;\; D \times (t_0, t_1), \\ T_2^* = 0 \quad \text{for} \;\; t = t_1, \end{cases}$$

$$\begin{cases} -(T_1^*)_t + L_1^* T_1^* = 0 \quad \text{in} \;\; D \times (t_0, t_1), \\ T_1^* = T_2^*(t_0) \quad \text{for} \;\; t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + T_2^* = 0 \quad \text{on} \;\; \Omega_0^{(1)} \times (t_0, t_1).$$

Functions T_2, $Q(t_1)$ are accepted as approximations to functions T, Q of the full solution for the Problem I at $t > t_1$, and $T_2(t_1) \cong T(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 2. Solve problem for S:

$$S_t + (\bar{U}, \mathbf{Grad}) S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function S is accepted as an approximate solution, and the function $S(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

Given $Q^{(k)}$ one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$

$$Q^{(k+1)} = Q^{(k)} - \gamma_k^{(j)}(\alpha(Q^{(k)} - Q^{(0)}) + T_2^*) \quad \text{on} \quad \Omega_0^{(j)} \times (t_{j-1}, t_j).$$

Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

$$\gamma_k^{(j)} = \frac{1}{2} \frac{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T - T_{obs}^{(j)})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

In view of the established properties of unique and dense solvability of the considered Problem Inv and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t})$.

Definition : Problem Inv 2 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\phi) < \epsilon$.

Proposition 1. Problem Inv 2 is uniquely and densely solvable. The solution of Problem A2 can be taken as an approximate solution of Problem Inv 2 for sufficiently small α.

5. Inverse and variational data assimilation problems, III

Problem Inv 3

Assume that the sea surface salinity (SSS), observed on a subset $\Omega^{(j)}$ of Ω, is denoted by $S_{obs} \equiv S_{obs}^{(j)}$ when $t \in (t_{j-1}, t_j), m_0^{(j)}$ is the characteristic function of this subset $(j = 1, 2, ..., J)$. Considering the boundary condition for S at $z = 0$ we write as:

$$-\nu_S \frac{\partial S}{\partial z} = Q \; at \; z = 0 \; on \; \Omega^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} S - \nu_S \frac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \; at \;\; z = 0 \; on \; (\Omega \backslash \Omega^{(j)}) \times (t_{j-1}, t_j),$$

where the function $Q \equiv Q^{(j)} (j = 1, 2, ..., J)$.

Let the functions $Q^{(j)}$ are "additional unknown functions" and we state the following inverse problem - **Problem Inv 3**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* $Q^{(j)}$, *such that,* $m_0^{(j)} (S - S_{obs}^{(j)}) = 0$ on $\Omega, j = 1, 2, ..., J$.

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(Q,\Phi) = \frac{1}{2}\int\limits_0^{\bar t}\int\limits_\Omega \alpha|Q-Q^{(0)}|^2 d\Omega dt + \Im_0(\Phi) = \sum_{j=1}^{J}\int\limits_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(Q) = \frac{1}{2}\int\limits_0^{\bar t}\int\limits_{\Omega_0(t)} m_0|S-S_{obs}|^2 d\Omega dt,$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\int\limits_{t_{j-1}}^{t_j}\int\limits_{\Omega^{(j)}} \alpha|Q-Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int\limits_{t_{j-1}}^{t_j}\int\limits_{\Omega^{(j)}} m_0^{(j)}|S-S_{obs}^{(j)}|^2 d\Omega dt.$$

Here $\alpha \geq 0$, is a "regularization"or "penalty"function ,that may be constant.
Data assimilation problem - **Problem A 3**: *find the solution* ϕ *of the Problem* I *and functions* $\{Q^{(j)}\}$, *such that, the cost functional is minimal on the set of the solutions.*
This problem for the case $\Omega^{(j)} \equiv \Omega (j = 1, 2, ..., J)$ has been studied and numerically solved by Agoshkov V.I., Parmuzin E.I. and Shutyaev V.P.[2008].

Definition : Problem Inv 3 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\phi) < \epsilon$.

Proposition 1. Problem Inv 3 is uniquely and densely solvable. The solution of Problem A3 can be taken as an approximate solution of Problem Inv 3 for sufficiently small α.

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. The method can be discribed as follows:

STEP 1. Solve problem for T:

$$T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function T is accepted as an approximate solution, and the function $T(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 2. We solve system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations. This system consists of equations for S_1, S_2, Q and system of adjoint equations:

$$\begin{cases} -(S_2^*)_t + L_2^* S_2^* = B^* m_0^{(1)}(S - S_{obs}^{(1)}) \quad \text{in} \quad D \times (t_0, t_1), \\ S_2^* = 0 \quad \text{for} \quad t = t_1, \end{cases}$$

$$\begin{cases} -(S_1^*)_t + L_1^* S_1^* = 0 \quad \text{in} \quad D \times (t_0, t_1), \\ S_1^* = S_2^*(t_0) \quad \text{for} \quad t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + S_2^* = 0 \quad \text{on} \quad \Omega_0^{(1)} \times (t_0, t_1).$$

Functions S_2, $Q(t_1)$ are accepted as approximations to functions S, Q of the full solution for the Problem I at $t > t_1$, and $S_2(t_1) \cong S(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

Iterative process

Given $Q^{(k)}$ one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$

$$Q^{(k+1)} = Q^{(k)} - \gamma_k^{(j)}(\alpha(Q^{(k)} - Q^{(0)}) + S_2^*) \quad \text{on} \;\; \Omega_0^{(j)} \times (t_{j-1}, t_j).$$

Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

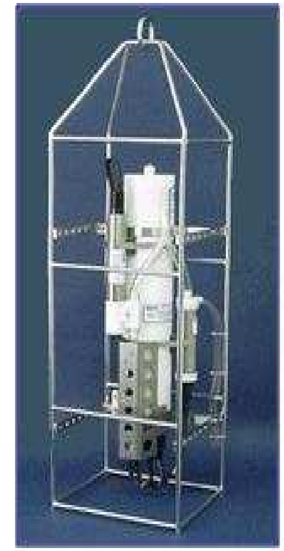
$$\gamma_k^{(j)} = \frac{1}{2} \frac{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (S - S_{obs}^{(j)})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (S_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

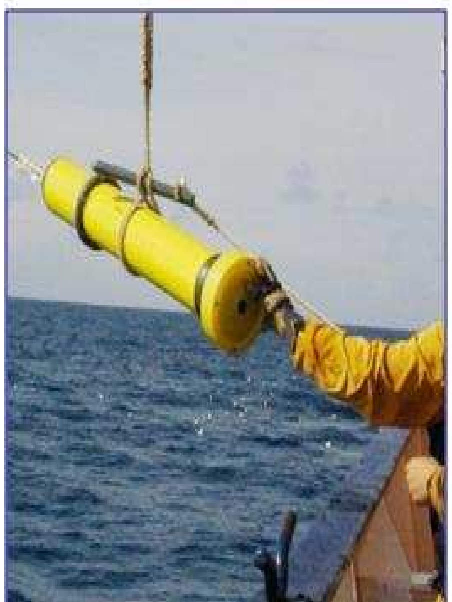
In view of the established properties of unique and dense solvability of the considered Problem Inv and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t}\,)$.

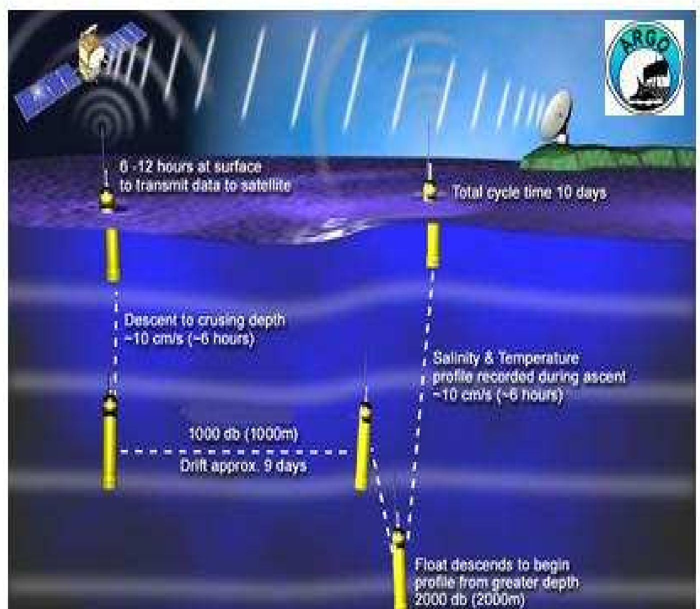
6. Information support of solving data assimilation problems

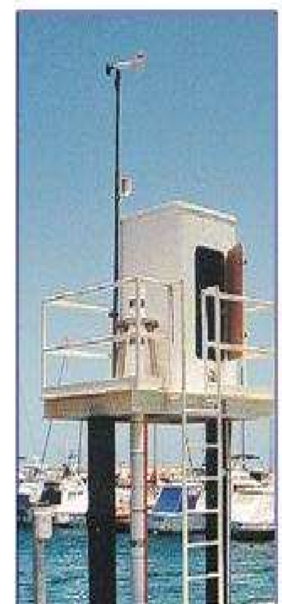
1. Data base (Lebedev S.A., 2005-2010)

Data Base «World Ocean – INM RAS»

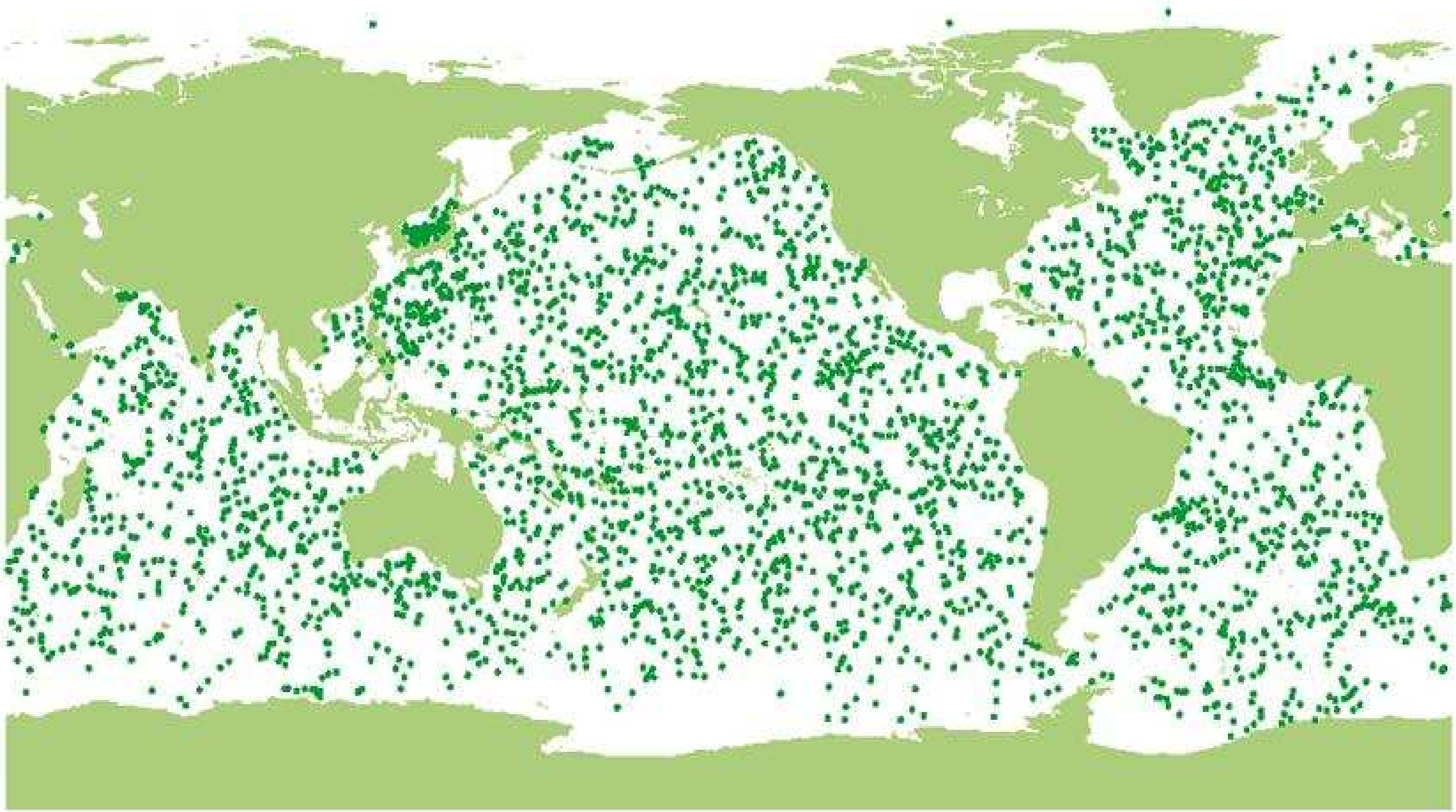








2. ARGO data (Zakharova N.B., 2009-2010)

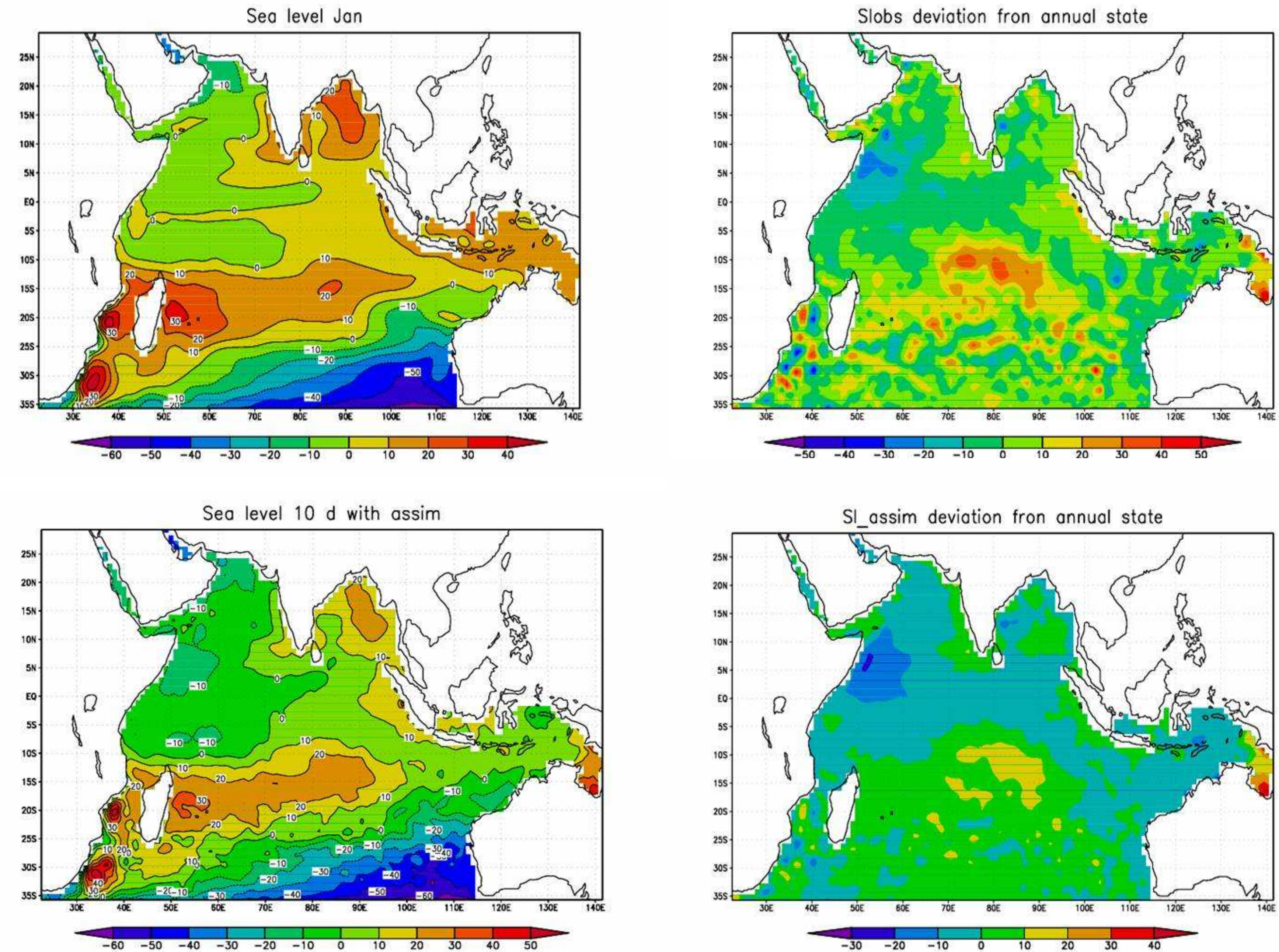


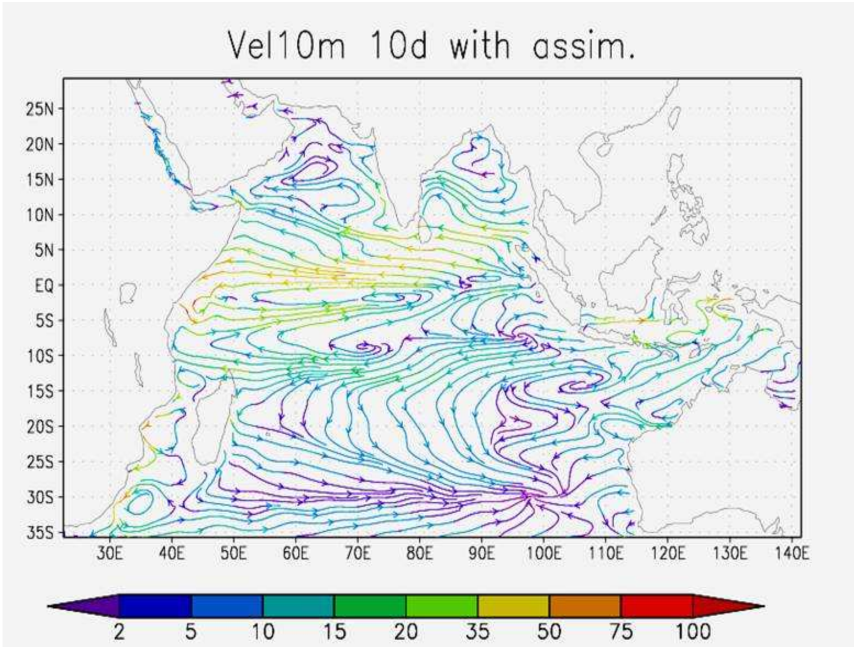
7. Numerical experiments

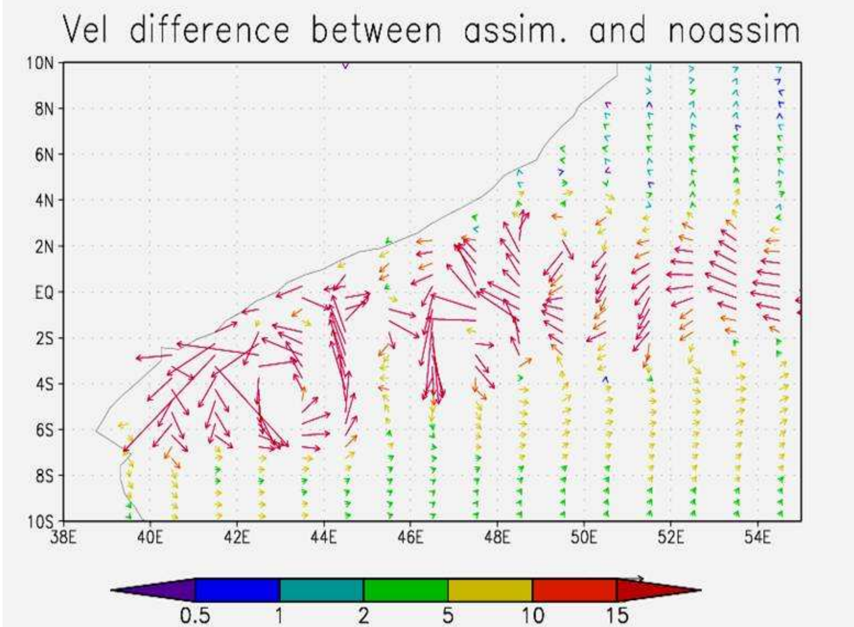
Problem Inv 1

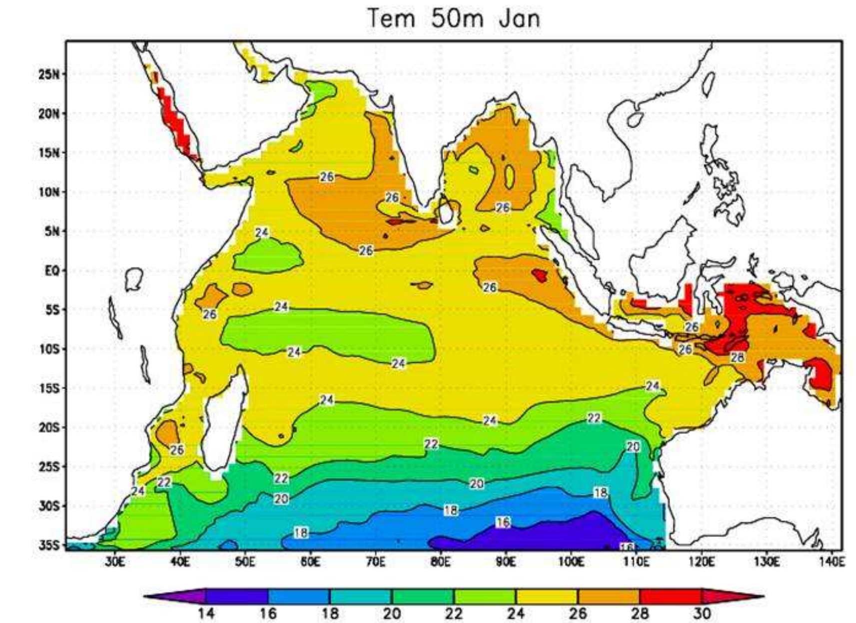
The object of simulation is the Indian Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 120x131x33 (latitude$\times$longitude$\times$depth); the first mesh point is the point with coordinates 22.5 E and 33.5 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour.
The data assimilation module to assimilate ξ_{obs} was included into the thermohydrodynamics model of the Indian Ocean. The time period taken in experiments is 1-10 days (January 2000).

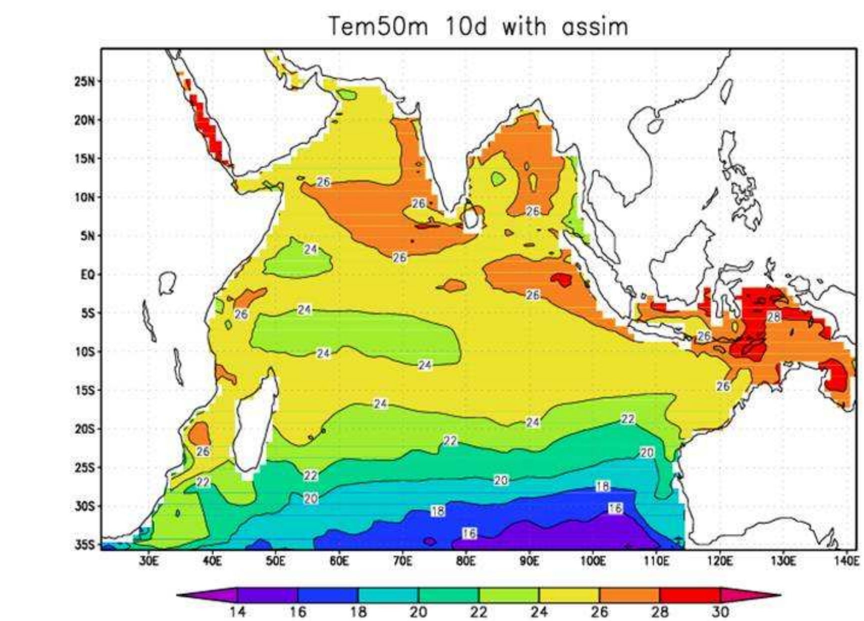
Numerical results with assimilation. The sea level function.




Vel10m 10d with assim.
2 5 10 15 20 35 50 75 100


Vel difference between assim. and noassim
0.5 1 2 5 10 15

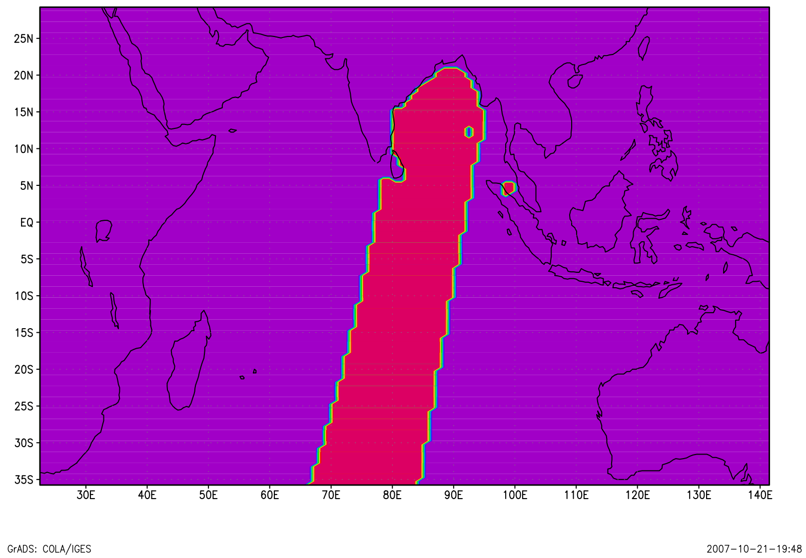

Tem 50m Jan
14 16 18 20 22 24 26 28 30


Tem50m 10d with assim
14 16 18 20 22 24 26 28 30

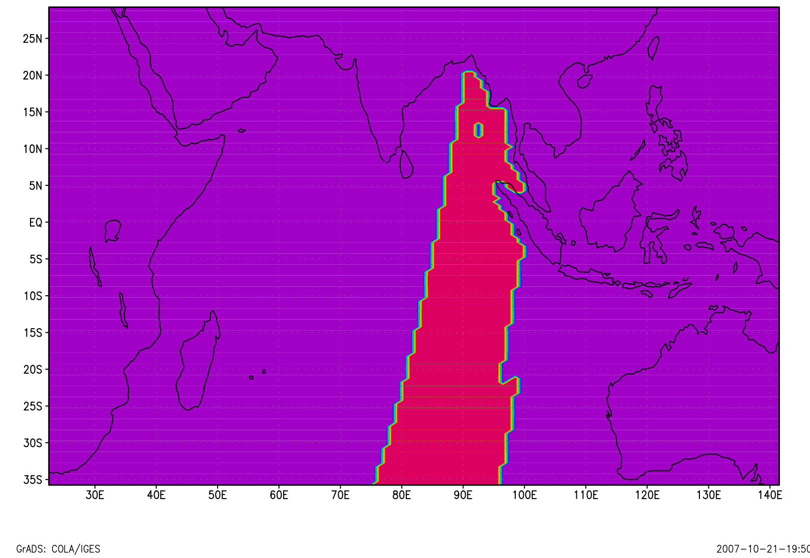
Problem Inv 2

The object of simulation is the Indian Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 120x131x33 (latitude×longitude×depth); the first mesh point is the point with coordinates 22.5 E and 33.5 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour.
The data of SST, were used for the construction of the function T_{obs} to be assimilated.
The observation data assimilation module to assimilate T_{obs} was included into the thermohydrodynamics model of the Indian Ocean. The longest time period taken in experiments was 3 months (start from January 2000).

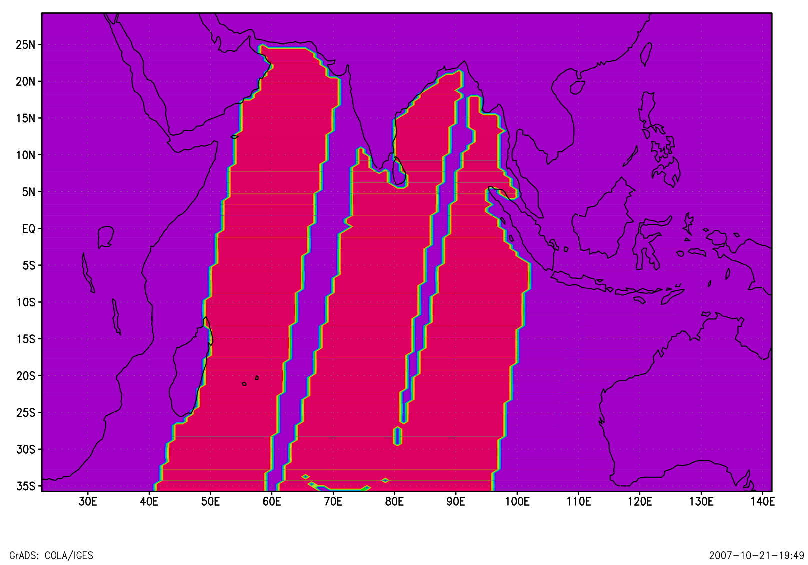
Observation data mask by hours



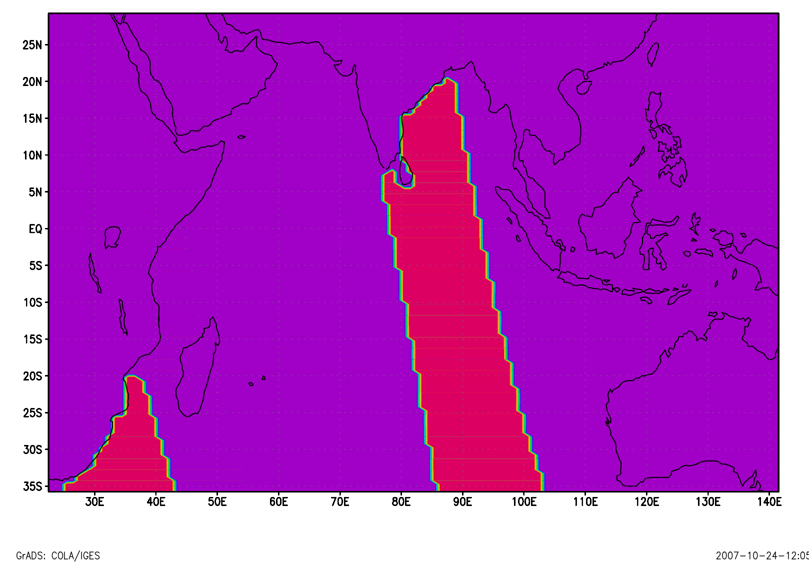
(a) 1



(b) 2

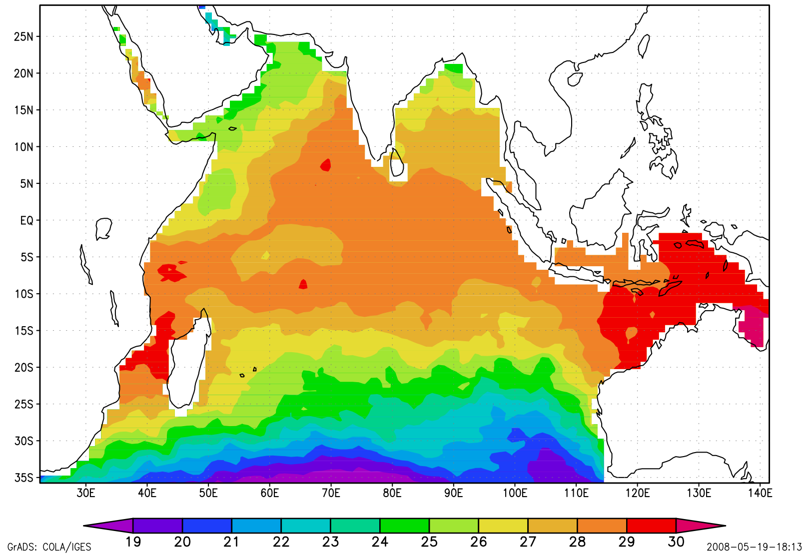


(c) 3

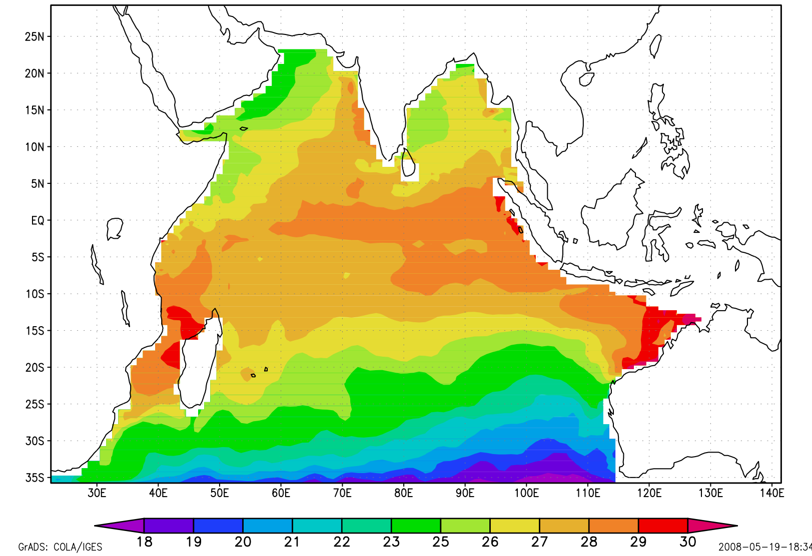


(d) 4

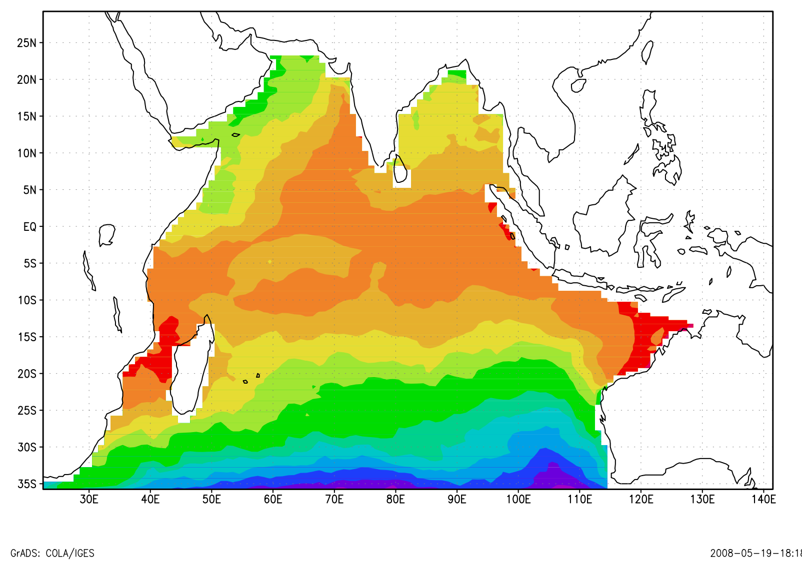
SST after 12th hours of simulations (start from 1 of January)



(a) The observation data



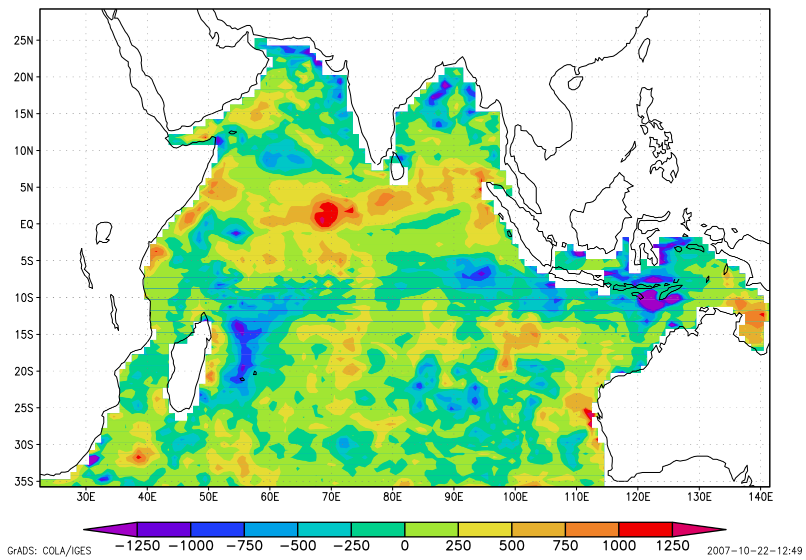
(b) SST obtaned without assimilation



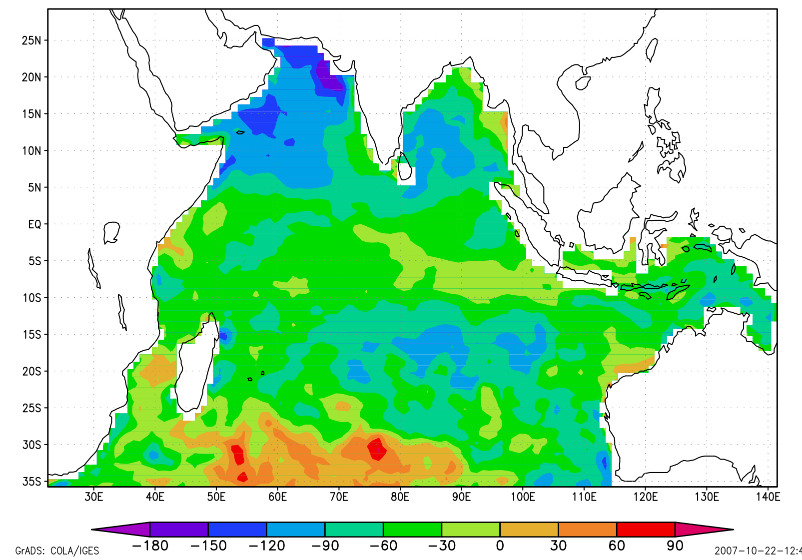
(c) SST calculated with assimilation

$\alpha = 10^{-6}$

Calculated by assimilation with $\alpha = 10^{-6}$ and climatic fluxes Q

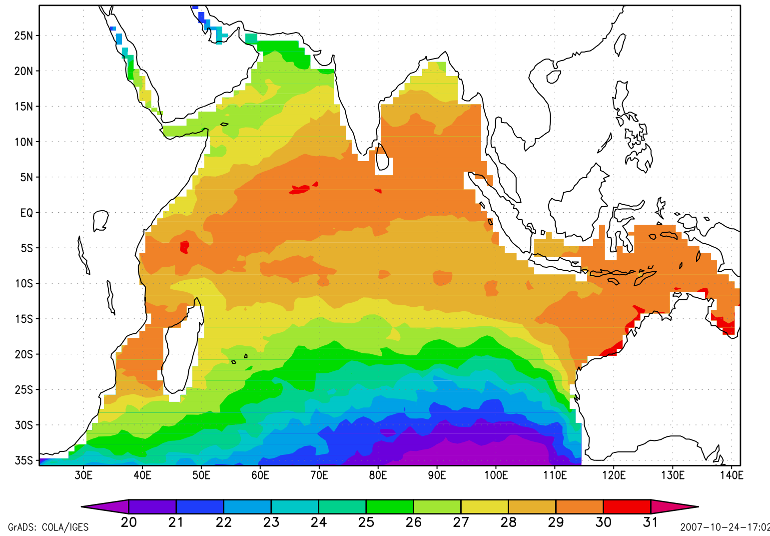


(a) Turbulent heat flux after assimilationon

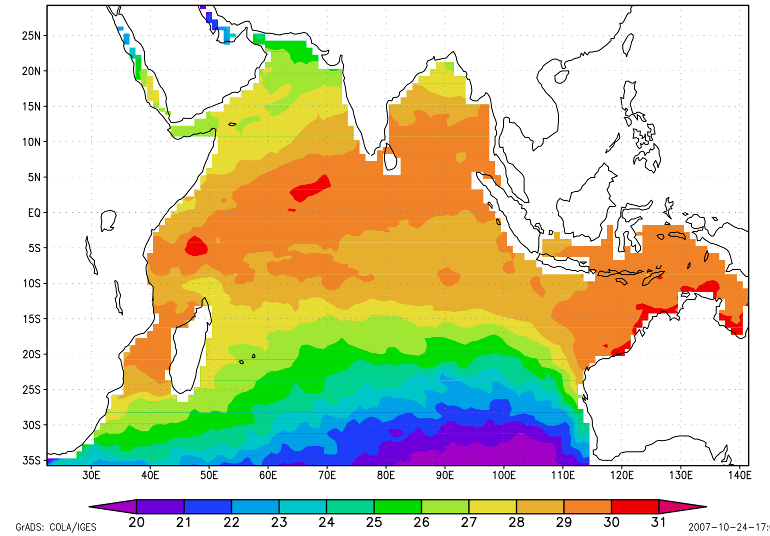


(b) Climatic mean heat flux

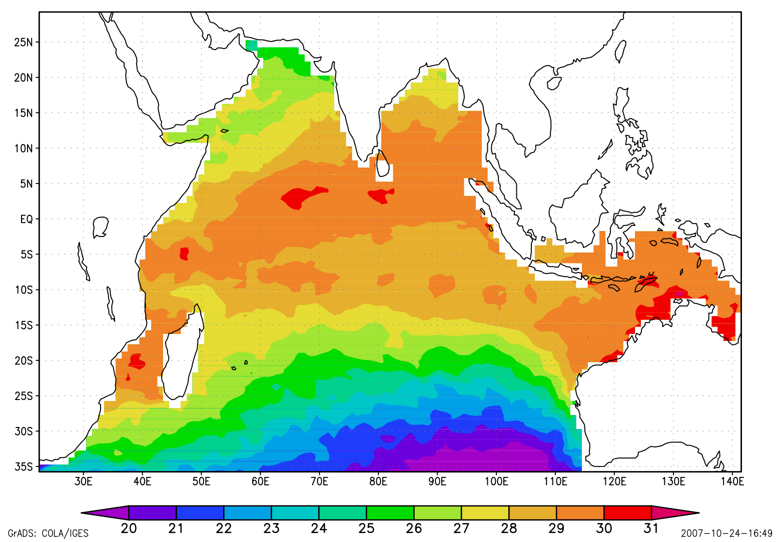
Calculated by assimilation with $\alpha = 10^{-6}$ and climatic fluxes Q



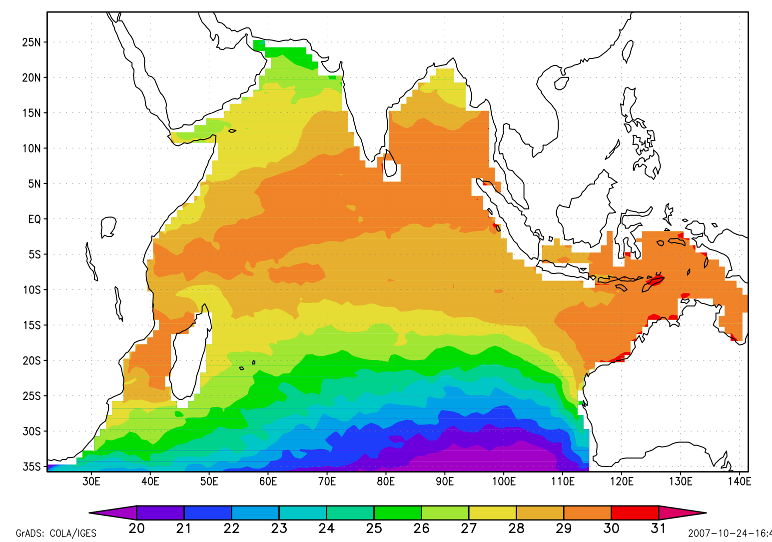
(a) Observation of SST 3 months and 9 hours



(b) Observation of SST 3 months and 3 days



(c) Assimilation of SST during 3 months and 9 hours prediction



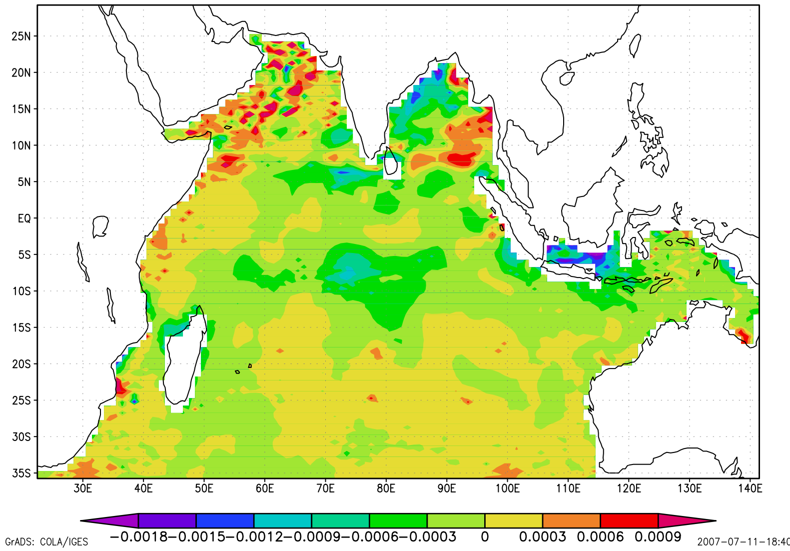
(d) Assimilation of SST 3 months and 3 days prediction

Problem Inv 3

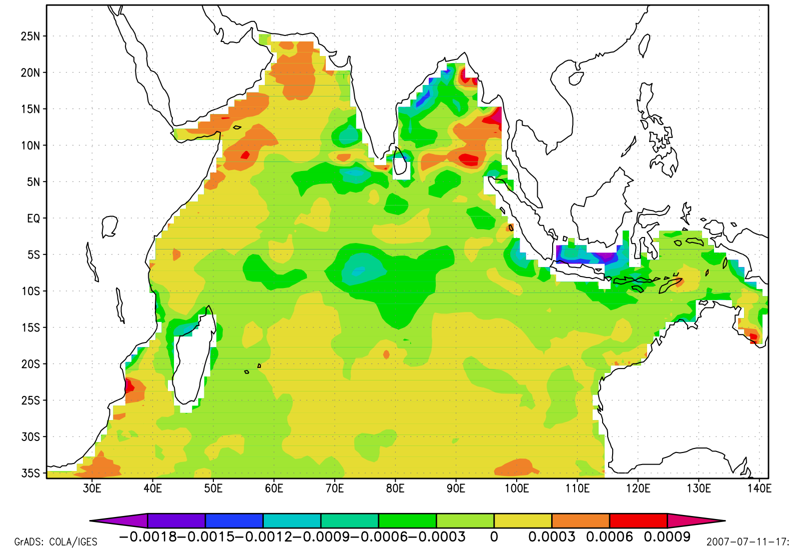
The object of simulation is the Indian Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 120x131x33 (latitude$\times$longitude$\times$depth); the first mesh point is the point with coordinates 22.5 E and 33.5 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour.

The data on salinity on the surface of the Indian Ocean for the period of January 2000 were used as S_{obs}. These data were calculated at each time moment on the considered grid (i.e., at each hour) based on the direct model. The salinity flux for the period of January 2000 was obtained by calculations from the direct model and used for $Q^{(0)}$.

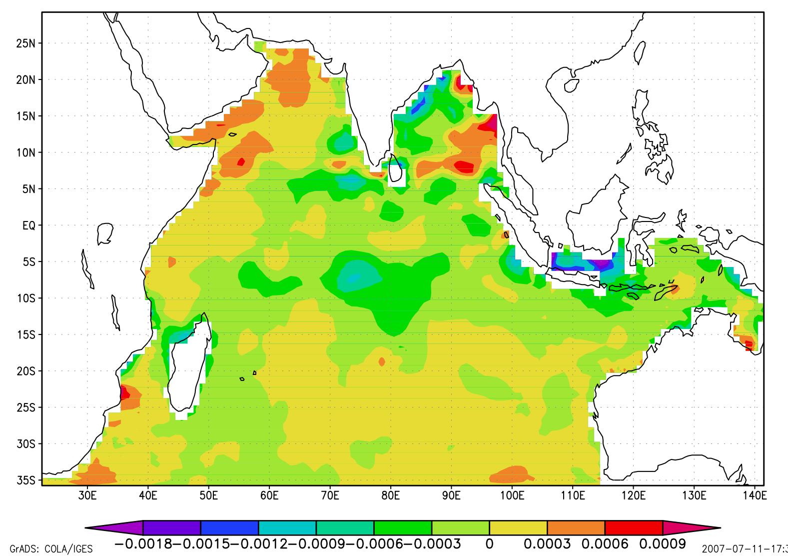
The observation data assimilation module to assimilate S_{obs} was included into the thermohydrodynamics model of the Indian Ocean. The longest time period taken in experiments was 15 days (January 2000).



(a) Calculation with assimilation for $\alpha = 10^{-5}$



(b) Calculation with assimilation for $\alpha = 10^{5}$

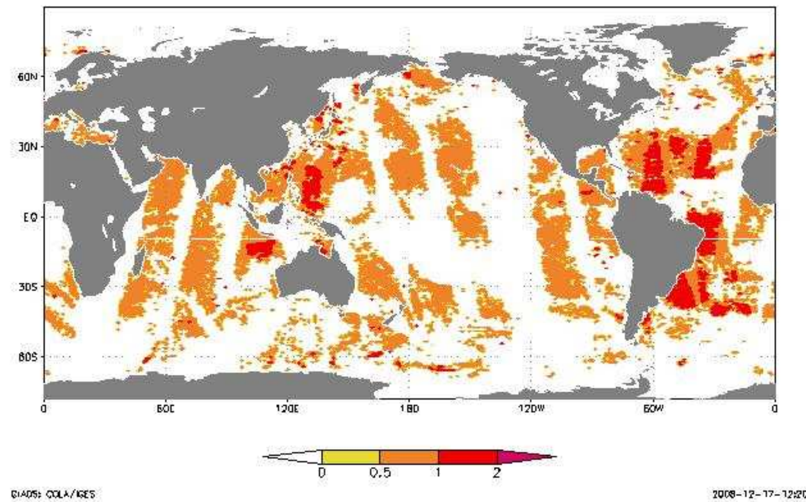


(c) Calculation by the model

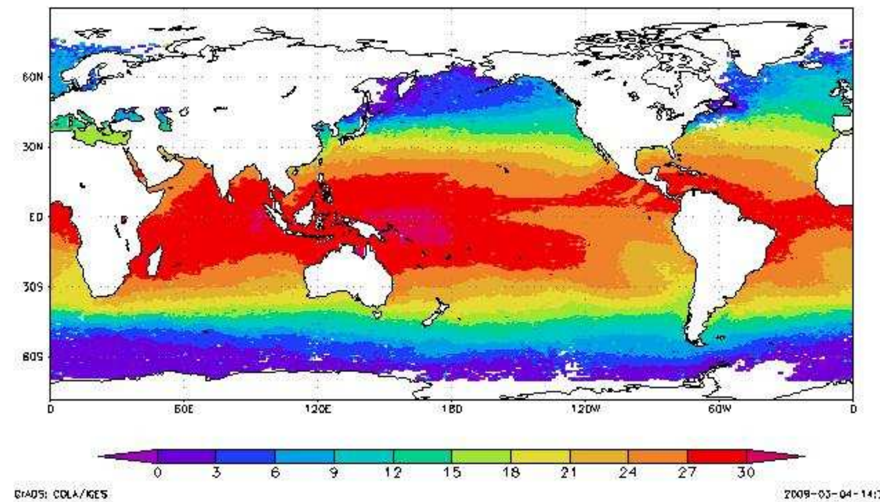
Problem Inv 4

The object of simulation is the World Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 360x227x40 (latitude×longitude×depth); the first mesh point is the point with coordinates 22.5 E and 78.25 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour.
The data of SST, were used for the construction of the function T_{obs} to be assimilated.
The observation data assimilation module to assimilate T_{obs} was included into the thermohydrodynamics model of the World Ocean. The longest time period taken in experiments was 1 months (start from January 2004).

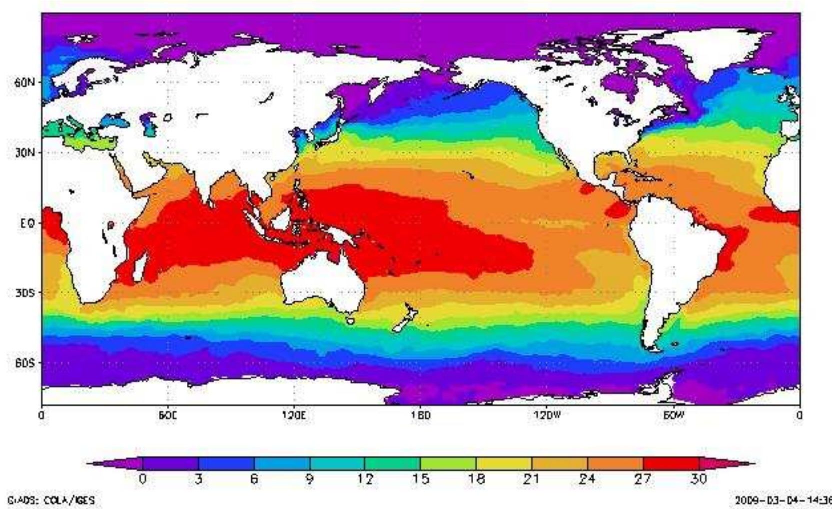
Calculated by assimilation with $\alpha = 10^{-6}$



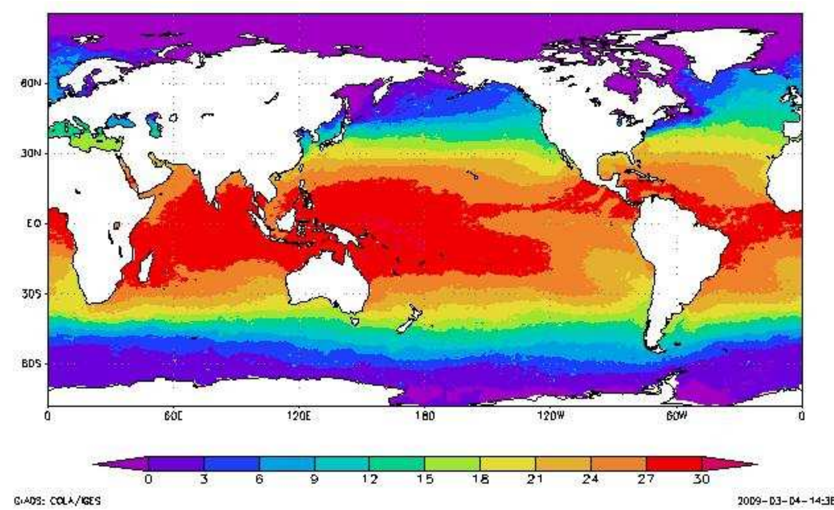
(a) Number of observation of SST (accumulated for 6 hours)



(b) Observation of SST



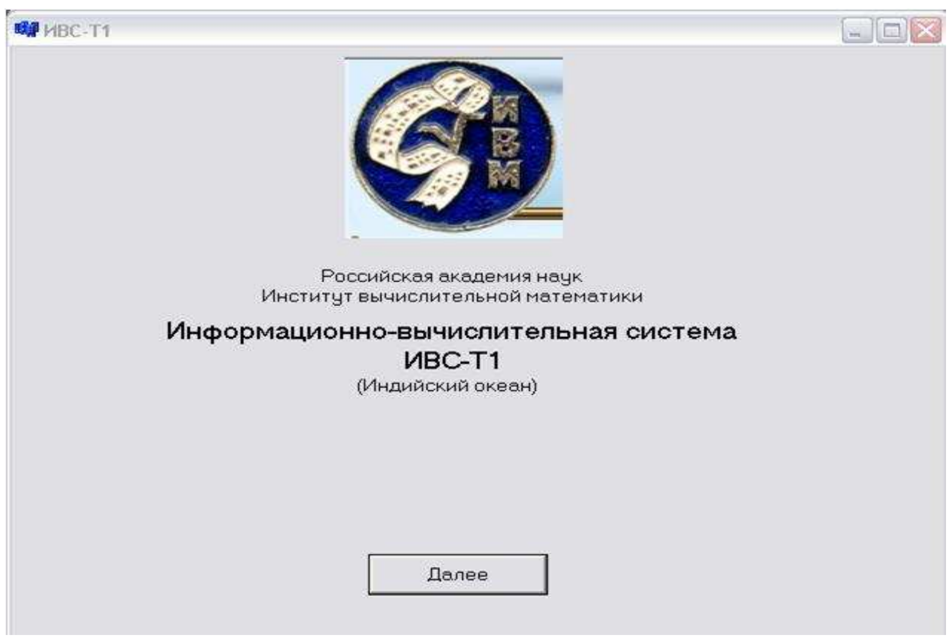
(c) Calculation by the model



(d) Assimilation of SST

Conclusion

- The class of inverse and corresponding variational data assimilation problems for the ocean dynamics mathematical models were formulated and studied.
- The obtained results are developing and applying now to variational "images"assimilation problems.
- The results obtained in INM RAS are the theoretical background for the construction of Informational-computational systems for variational data assimilation into the ocean circulation model.




Thank You!