Tensorisation and Convolution

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1 Tensorisation

Let the vector $y \in \mathbb{C}^n$ represent the grid values of a function in $[0,1]$:

$$y_\mu = f\left(\frac{\mu+1}{n}\right) \qquad (0 \leq \mu \leq n-1)\,.$$

Choose, e.g., $n = 2^d$. Note that

$$\mathbb{C}^n \cong \mathbf{V} := \bigotimes_{j=1}^{d} \mathbb{C}^2.$$

Isomorphism by binary integer representation:
$\mu = \sum_{j=1}^{d} \mu_j 2^{j-1}$ with $\mu_j \in \{0,1\}$, i.e.,

$$y_\mu = \mathbf{v}_{\mu_1\mu_2\ldots\mu_{d-1}\mu_d} \qquad \left(0 \leq \mu \leq n-1, \quad 0 \leq \mu_j \leq 1\right).$$

See Oseledets-Tyrtyshnikov.

Algebraic Function Compression (black-box procedure)

1) Tensorisation: $y \in \mathbb{C}^n \longmapsto \mathbf{v} \in \mathbf{V}$ (storage size: $n = 2^d$)
2) Apply the tensor truncation: $\mathbf{v} \longmapsto \mathbf{v}_\varepsilon$
3) Observation: often the data size decreases from $n = 2^d$ to $O(d) = O(\log n)$.

This observation can be proved under natural conditions of the underlying function (Grasedyck 2010).

2 Convolution

2.1 Definition

Convolution for vectors $v, w \in \mathbb{C}^n$ is defined by

$$u = v \star w \quad \text{with } u_k = \sum_{\ell=0}^{k} v_\ell w_{k-\ell} \qquad (0 \le k \le 2n-2)\,.$$

Problem:

Let $v, w \in \mathbb{C}^n$ correspond to tensors $\mathbf{v}, \mathbf{w} \in \mathbf{V}$. The computation of $u = v \star w$ corresponding to $\mathbf{u} \in \mathbf{V}$ is abbreviated by

$$\mathbf{u} := \mathbf{v} \star \mathbf{w}.$$

Aim:

The cost of the computation $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{u} = \mathbf{v} \star \mathbf{w}$ should correspond to the data sizes of $\mathbf{v}, \mathbf{w}$.
That means, it is not allowed to use $(\mathbf{v}, \mathbf{w}) \mapsto (v, w) \mapsto u = v \star w \mapsto \mathbf{u}$, since then the cost is $> n$.

2.2 Multidimensional Convolution

We recall the convolution of multivariate functions:

$$\left(\bigotimes_{j=1}^{d} f_j\right) \star \left(\bigotimes_{j=1}^{d} g_j\right) = \bigotimes_{j=1}^{d} \left(f_j \star g_j\right),$$

i.e., multivariate convolution in d variables is equivalent to d 1D convolutions.

Similarly, the convolution of d-dimensional grid functions $\mathbf{v}, \mathbf{w} \in \bigotimes_{j=1}^{d} \mathbb{C}^{n_j}$ defined by

$$u_{\mathbf{k}} = \sum_{\mathbf{0} \leq \boldsymbol{\ell} \leq \mathbf{k}} v_{\boldsymbol{\ell}} w_{\mathbf{k}-\boldsymbol{\ell}}$$

satisfies

$$\left(\bigotimes_{j=1}^{d} v_j\right) \star \left(\bigotimes_{j=1}^{d} w_j\right) = \bigotimes_{j=1}^{d} \left(v_j \star w_j\right).$$

- The tensorisation isomorphism maps the 1D vectors $v, w \in \mathbb{C}^n$ into order-d tensors in $\mathbf{V} = \bigotimes_{j=1}^{d} \mathbb{C}^2$.

Question:

Can we reduce the convolution $\mathbf{u} = \mathbf{v} \star \mathbf{w}$ to d convolutions for each direction j separately?

- First idea:
  Maybe not, since the d directions are of artificial nature, whereas the true vector is one-dimensional.

- Second idea:
  But we may try!

Closer look:

$v, w \in \mathbb{C}^n$, choice of index range: $v = (v_i)_{i=0}^{n-1}$, $w = (w_i)_{i=0}^{n-1}$,

definition:

$$u_k = \sum_{\ell=0}^{k} v_\ell w_{k-\ell} \qquad (0 \le k \le 2n-2).$$

$\Rightarrow u \in \mathbb{C}^{2n-1}$.

In particular, $v, w \in \mathbb{C}^2 \Rightarrow v \star w \in \mathbb{C}^3$.

Hence, a formula like

$$\left(\bigotimes_{j=1}^{d} v_j\right) \star \left(\bigotimes_{j=1}^{d} w_j\right) = \bigotimes_{j=1}^{d} \underbrace{\left(v_j \star w_j\right)}_{\in \mathbb{C}^3}$$

$$\text{for} \quad \mathbf{v} = \bigotimes_{j=1}^{d} v_j, \mathbf{w} = \bigotimes_{j=1}^{d} w_j \in \mathbf{V} = \bigotimes_{j=1}^{d} \mathbb{C}^2$$

does not make sense because the wrong formats.

Nevertheless, ... (*Preprint 48, MPI Leipzig, 2010)*

2.3 Guiding Model Problem

Decimal Format of integers:

$$n = \sum_{k \geq 0} n_k 10^k \qquad \text{with } n_k \in \{0, 1, \ldots, 9\}.$$

Digit-wise addition of $s = 178 + 245$:

$$\begin{array}{ccc} 1 & 7 & 8 \\ 2 & 4 & 5 \\ \hline 3 & (11) & (13) \end{array}$$

leads to $s = 3(11)(13)$ in the **Generalised Format**

$$n = \sum_{k \geq 0} n_k 10^k \qquad \text{with } n_k \in \mathbb{N}_0.$$

- The representation in the generalised format is not unique.
- Conversion into the decimal format by carry-over:

$$\ldots n_{k+1} \left(a_k + 10 b_k\right) \ldots = \ldots \left(n_{k+1} + b_k\right) a_k \ldots$$

3 Analysis of the Problem

3.1 Notations

Isomorphism Φ_n (interpretation of the tensor as vector)

$$\Phi_n : \mathbf{V} := \bigotimes_{j=1}^{d} \mathbb{C}^2 \to \mathbb{C}^n \qquad \text{for } n = 2^d$$

defined by

$$\Phi_n : \mathbf{v} \in \mathbf{V} \mapsto v = (v_k)_{k=0}^{n-1} \in \mathbb{C}^n \text{ with } v_k = \mathbf{v}_{i_1 i_2 \ldots i_d},$$
$$\text{where } k = \sum_{j=1}^{d} i_j 2^{j-1},\ i_j \in \{0,1\}.$$

3.2 $\mathbb{C}^n$ and ℓ_0

Set of *infinite sequences* with only finitely many nonzero entries:

$$\ell_0 := \{(a_i)_{i\in\mathbb{N}_0} : a_i = 0 \text{ for almost all } i \in \mathbb{N}_0\}.$$

The embedding of $\mathbb{C}^n$ into ℓ_0 is defined by

$$\lambda_n : \mathbb{C}^n \to \ell_0, \qquad v \in \mathbb{C}^n \mapsto a = (a_i)_{i\in\mathbb{N}_0} \in \ell_0 \quad \text{with } a_i := \begin{cases} v_i & \text{for } 0 \le i \le n-1, \\ 0 & \text{for } i \ge n. \end{cases}$$

Degree of $a \in \ell_0$ is defined by

$$\deg(a) := \max\{i \in \mathbb{N}_0 : a_i \neq 0\}.$$

Obviously, λ_n maps $\mathbb{C}^n$ into $a \in \ell_0$ with $\deg(a) \le n-1$.

For convenience we suppress the notation λ_n and identify $\mathbb{C}^n$ with the subset of sequences of degree $\le n-1$:

$$\mathbb{C}^n \subset \ell_0$$

Shift operator S^m $(m \in \mathbb{Z})$:

$$b = S^m(a) \text{ has entries } b_i = \begin{cases} a_{i-m} & \text{if } m \le i, \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Tensor space $\bigotimes_{j=1}^{d} \ell_0$

The embedding $\mathbb{C}^n \subset \ell_0$ (in particularly, $\mathbb{C}^2 \subset \ell_0$) leads to an embedding

$$\bigotimes_{j=1}^{d} \mathbb{C}^2 \subset \bigotimes_{j=1}^{d} \ell_0.$$

The mapping $\Phi_n : \bigotimes_{j=1}^{d} \mathbb{C}^2 \to \mathbb{C}^n \subset \ell_0$ can be extended to

$$\Phi : \bigotimes_{j=1}^{d} \ell_0 \to \ell_0 \,,$$

$$\text{by } \mathbf{v} \mapsto a = (a_i)_{i\in\mathbb{N}_0} \qquad \text{with } a_k = \sum_{\substack{i_1,\ldots,i_d \in \mathbb{N}_0 \\ k=\sum_{j=1}^{d} i_j 2^{j-1}}} \mathbf{v}_{i_1 i_2 \ldots i_d}.$$

Note that

- k may possess multiple representations $\sum_{j=1}^{d} i_j 2^{j-1}$.
- $\Phi : \bigotimes_{j=1}^{d} \ell_0 \to \ell_0$ is not injective.
- Only if $\mathbf{v} \in \bigotimes_{j=1}^{d} \mathbb{C}^2 \subset \bigotimes_{j=1}^{d} \ell_0$, the indices i_j are restricted to $\{0, 1\}$ and each integer k has exactly one representation $k = \sum_{j=1}^{d} i_j 2^{j-1}$. Then the definition coincides with the original definition of Φ_n.

3.4 Equivalence Relation

By means of Φ_n we define an equivalence relation in $\bigotimes_{j=1}^d \ell_0$ via

$$\mathbf{v} \sim \mathbf{w} \quad \text{if and only if } \Phi_n(\mathbf{v}) = \Phi_n(\mathbf{w}) \qquad (\mathbf{v}, \mathbf{w} \in \bigotimes_{j=1}^d \ell_0).$$

If $\deg(\mathbf{v}) := \deg(\Phi_n(\mathbf{v})) \leq n-1$, there is a unique $\hat{\mathbf{v}} \in \bigotimes_{j=1}^d \mathbb{C}^2 \subset \bigotimes_{j=1}^d \ell_0$ with $\mathbf{v} \sim \hat{\mathbf{v}}$.

The shift operator can be used together with the tensor product.

LEMMA: Let $m = \sum_{j=1}^d m_j 2^{j-1}$. Then

$$\Phi_n \left(\bigotimes_{j=1}^d S^{m_j} v^{(j)} \right) = S^m \Phi_n \left(\bigotimes_{j=1}^d v^{(j)} \right).$$

So far, the action of S is defined for vectors of ℓ_0 only. For tensors, we set

$$S^m \bigotimes_{j=1}^d v^{(j)} := \left(S^m v^{(1)} \right) \otimes \bigotimes_{j=2}^d v^{(j)},$$

Since

$$v, w \in \mathbb{C}^n \Rightarrow u := v \star w \in \mathbb{C}^{2n-1} \subset \mathbb{C}^{2n}$$

the tensor representation of u requires a tensor of order $d+1$!

The definitions of Φ and, thereby, of the equivalent relation does not need a fixed d.

Since

$$\mathbf{\Phi}\left(\mathbf{v} \otimes \binom{1}{0}\right) = \mathbf{\Phi}(\mathbf{v})$$

for any tensor $\mathbf{v} \in \bigotimes_{j=1}^{k} \ell_0$, one obtains

$$\left.\begin{array}{lll} \mathbf{v} \otimes \binom{1}{0} & \sim & \mathbf{v} \\ \mathbf{v} \otimes \binom{0}{1} & \sim & S^{2^k}\mathbf{v} \end{array}\right\} \quad \text{for all } \mathbf{v} \in \bigotimes_{j=1}^{k} \ell_0.$$

3.5 Convolution in ℓ_0 and $\mathbb{C}^n$

For the convolution of vectors $v, w \in \mathbb{C}^n$, we consider $\mathbb{C}^n$ as embedded in ℓ_0. The convolution in ℓ_0 is given by

$$a, b \in \ell_0 \quad \mapsto \quad c := a \star b \in \ell_0 \quad \text{with } c_k := \sum_{j=0}^{k} a_j b_{k-j} \ (k \in \mathbb{N}_0)$$

The vector space ℓ_0 is isomorphic to the vector space $\mathbb{P}$ of *polynomials* of finite, but arbitrary degree:

$$\pi : \ell_0 \to \mathbb{P} \qquad \text{with } \pi(a) = \sum_{j=0}^{\infty} a_j x^j.$$

According to the embedding $\mathbb{C}^n \subset \ell_0$, we also use π as mapping from $\mathbb{C}^n$ into $\mathbb{P}$ (onto all polynomials of degree $\leq n-1$).

The convolution in ℓ_0 corresponds to the (pointwise) multiplication in $\mathbb{P}$:

$$c := a \star b \in \ell_0 \quad \Leftrightarrow \quad \pi(c) = \pi(a)\pi(b).$$

3.6 $\bigotimes_{\delta=1}^{d} \mathbb{C}^2$, $\bigotimes_{\delta=1}^{d} \ell_0$, and polynomials

Consider an elementary tensor $\mathbf{v} = \bigotimes_{\delta=1}^{d} v^{(\delta)}$ ($v^{(\delta)} \in \mathbb{C}^2$) and the corresponding vector $v = \Phi_n(\mathbf{v}) \in \mathbb{C}^n$. Applying π to v, we obtain the polynomial $p := \pi(v)$ with $p(x) := \sum_{j=0}^{n-1} v_j x^j$.

For ease of notation, we shall write π instead of $\pi \circ \Phi_n$, i.e.,

$$\pi : \bigotimes_{\delta=1}^{d} \mathbb{C}^2 \to \mathbb{P}.$$

The connection between $\mathbf{v} = \bigotimes_{\delta=1}^{d} v^{(\delta)}$ and $v = \Phi_n(\mathbf{v})$ on the side of polynomials is given by

$$p = \pi(\mathbf{v}): \quad p(x) = \prod_{\delta=1}^{d} p_\delta(x^{2^{\delta-1}}) \qquad \text{with } p_\delta := \pi(v^{(\delta)}),$$

where the linear polynomials $p_\delta(\xi) = v_0^{(\delta)} + v_1^{(\delta)}\xi$ are substituted by $\xi = x^{2^{\delta-1}}$. The definition of $\pi(\mathbf{v})$ from above can be extended from $\bigotimes_{\delta=1}^{d} \mathbb{C}^2$ to $\bigotimes_{\delta=1}^{d} \ell_0$.

We note that $\mathbf{v} \sim \mathbf{w} \Leftrightarrow \pi(\mathbf{v}) = \pi(\mathbf{w})$ for $\mathbf{v}, \mathbf{w} \in \bigotimes_{\delta=1}^{d} \ell_0$.

3.7 Convolution of Tensors

The elementary tensors

$$\mathbf{v} = \bigotimes_{\delta=1}^{d} v^{(\delta)}, \qquad v^{(\delta)} \in \mathbb{C}^2,$$
$$\mathbf{w} = \bigotimes_{\delta=1}^{d} w^{(\delta)}, \qquad w^{(\delta)} \in \mathbb{C}^2,$$

represent the vectors $v = \Phi_n(\mathbf{v})$ and $w = \Phi_n(\mathbf{w})$ in $\mathbb{C}^n$.
Therefore the convolution $\mathbf{v} \star \mathbf{w}$ must be defined such that

$$\Phi_n\left(\mathbf{v} \star \mathbf{w}\right) = \Phi_n(\mathbf{v}) \star \Phi_n(\mathbf{w}).$$

Note that this equation fixes only the equivalence class of $\mathbf{v} \star \mathbf{w}$.

LEMMA. The convolution of $\mathbf{v} = \bigotimes_{\delta=1}^{d} v^{(\delta)}$ and $\mathbf{w} = \bigotimes_{\delta=1}^{d} w^{(\delta)} \quad (v^{(\delta)}, w^{(\delta)} \in \ell_0)$ yields

$$\mathbf{v} \star \mathbf{w} = \bigotimes_{\delta=1}^{d} \left(v^{(\delta)} \star w^{(\delta)} \right), \qquad \text{where } v^{(\delta)} \star w^{(\delta)} \in \ell_0.$$

PROOF. Set $p := \pi(v)$, $p_\delta := \pi(v^{(\delta)})$ and $q := \pi(w)$, $q_\delta := \pi(w^{(\delta)})$. By definition, $\mathbf{u} := \bigotimes_{\delta=1}^{d} \left(v^{(\delta)} \star w^{(\delta)} \right)$ corresponds to the vector $u = \Phi(\mathbf{u})$ associated with the polynomial

$$\begin{aligned}
\pi(u)(x) &= \prod_{\delta=1}^{d} \pi(v^{(\delta)} \star w^{(\delta)})(x^{2^{\delta-1}}) = \prod_{\delta=1}^{d} \pi(v^{(\delta)})(x^{2^{\delta-1}}) \cdot \pi(w^{(\delta)})(x^{2^{\delta-1}}) \\
&= \prod_{\delta=1}^{d} p_\delta(x^{2^{\delta-1}}) q_\delta(x^{2^{\delta-1}}) = \left(\prod_{\delta=1}^{d} p_\delta(x^{2^{\delta-1}}) \right) \left(\prod_{\delta=1}^{d} q_\delta(x^{2^{\delta-1}}) \right) \\
&= p(x)q(x) = \pi(v \star w)(x),
\end{aligned}$$

which proves $u = v \star w$.

3.8 Induction start: Convolution in $\mathbb{C}^2$

LEMMA. The convolution of $v = \binom{\alpha}{\beta}, w = \binom{\gamma}{\delta} \in \mathbb{C}^2 = \bigotimes_{j=1}^{1} \mathbb{C}^2$ yields

$$\binom{\alpha}{\beta} \star \binom{\gamma}{\delta} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta + \beta\gamma \\ \beta\delta \\ 0 \end{pmatrix} = \binom{\alpha\gamma}{\alpha\delta+\beta\gamma} + S^2 \binom{\beta\delta}{0}$$

$$= \boldsymbol{\Phi}(\mathbf{v}) \quad \text{with } \mathbf{v} := \binom{\alpha\gamma}{\alpha\delta+\beta\gamma} \otimes \binom{1}{0} + \binom{\beta\delta}{0} \otimes \binom{0}{1} \in \bigotimes_{j=1}^{2} \mathbb{C}^2.$$

Furthermore, the shifted vector $S^1\left(\binom{\alpha}{\beta} \star \binom{\gamma}{\delta}\right)$ has the tensor representation

$$S^1\left(\binom{\alpha}{\beta} \star \binom{\gamma}{\delta}\right) = \begin{pmatrix} 0 \\ \alpha\gamma \\ \alpha\delta + \beta\gamma \\ \beta\delta \end{pmatrix}$$

$$= \boldsymbol{\Phi}(\mathbf{v}) \text{ with } \mathbf{v} := \binom{0}{\alpha\gamma} \otimes \binom{1}{0} + \binom{\alpha\delta+\beta\gamma}{\beta\delta} \otimes \binom{0}{1} \in \bigotimes_{j=1}^{2} \mathbb{C}^2.$$

3.9 Induction: $\delta - 1 \mapsto \delta$

LEMMA. Assume $\mathbf{v}, \mathbf{w} \in \bigotimes_{j=1}^{\delta-1} \mathbb{C}^2$ and $x = \binom{\alpha}{\beta}, y = \binom{\gamma}{\delta} \in \mathbb{C}^2$. Let the convolution result of $\mathbf{v}, \mathbf{w}$ be known:

$$\mathbf{v} \star \mathbf{w} \sim \mathbf{a} = \mathbf{a}' \otimes \binom{1}{0} + \mathbf{a}'' \otimes \binom{0}{1} \in \bigotimes_{j=1}^{\delta} \mathbb{C}^2.$$

Then, convolution of the tensors $\mathbf{v} \otimes x$ and $\mathbf{w} \otimes y$ yields

$$(\mathbf{v} \otimes x) \star (\mathbf{w} \otimes y) \sim \mathbf{u} = \mathbf{u}' \otimes \binom{1}{0} + \mathbf{u}'' \otimes \binom{0}{1} \in \bigotimes_{j=1}^{\delta+1} \mathbb{C}^2$$

with

$$\mathbf{u}' = \mathbf{a}' \otimes \binom{\alpha\gamma}{\alpha\delta+\beta\gamma} + \mathbf{a}'' \otimes \binom{0}{\alpha\gamma} \in \bigotimes_{j=1}^{\delta} \mathbb{C}^2,$$
$$\mathbf{u}'' = \mathbf{a}' \otimes \binom{\beta\delta}{0} + \mathbf{a}'' \otimes \binom{\alpha\delta+\beta\gamma}{\beta\delta} \in \bigotimes_{j=1}^{\delta} \mathbb{C}^2.$$

PROOF. Componentwise convolution implies that

$$(\mathbf{v}\otimes x)\star(\mathbf{w}\otimes y)\sim(\mathbf{v}\star\mathbf{w})\otimes z \qquad \text{with } z:=x\star y\in\mathbb{C}^3\subset\ell_0.$$

$\mathbf{v}\star\mathbf{w}\sim\mathbf{a}'\otimes\binom{1}{0}+\mathbf{a}''\otimes\binom{0}{1}$ leads to

$$(\mathbf{v}\star\mathbf{w})\otimes z\sim\left(\mathbf{a}'+S^{2^{\delta-1}}\mathbf{a}''\right)\otimes z.$$

The rule for the shift ('carry over') shows that

$$S^{2^{\delta-1}}\mathbf{a}''\otimes z=S^{2^{\delta-1}}(\mathbf{a}''\otimes z)\sim\mathbf{a}''\otimes(Sz).$$

Insert the result for $z=x\star y$:

$$\mathbf{a}'\otimes z\sim\mathbf{a}'\otimes\binom{\alpha\gamma}{\alpha\delta+\beta\gamma}\otimes\binom{1}{0}+\mathbf{a}'\otimes\binom{\beta\delta}{0}\otimes\binom{0}{1},$$

$$\left(S^{2^{\delta-1}}\mathbf{a}''\right)\otimes z\sim\mathbf{a}''\otimes(Sz)\sim\mathbf{a}''\otimes\binom{0}{\alpha\gamma}\otimes\binom{1}{0}+\mathbf{a}''\otimes\binom{\alpha\delta+\beta\gamma}{\beta\delta}\otimes\binom{0}{1}.$$

Summation of both identities yields the assertion of the lemma.

4 TT or Hierarchical Tensor Format

4.1 Format

The TT format $\mathcal{T}_{\mathbf{r}}$ with ranks $\mathbf{r} = (r_1 = 2, r_2, \ldots, r_d)$ is characterised algebraically as follows.

For $1 \leq \delta \leq d$ there are subspaces

$$U_\delta \subset \bigotimes_{j=1}^{\delta} \mathbb{C}^2$$

with

$$U_1 = \mathbb{C}^2$$

and

$$\dim(U_\delta) = r_\delta, \quad U_\delta \subset U_{\delta-1} \otimes \mathbb{C}^2 \qquad (2 \leq \delta \leq d).$$

A tensor $\mathbf{v} \in \bigotimes_{j=1}^{d} \mathbb{C}^2$ is represented if

$$\mathbf{v} \in U_d.$$

4.2 Mappings $\varphi'_\delta, \varphi''_\delta$

We recall the expression $(\mathbf{v} \otimes x) \star (\mathbf{w} \otimes y) \sim \mathbf{u} = \mathbf{u}' \otimes \binom{1}{0} + \mathbf{u}'' \otimes \binom{0}{1}$ with

$$\mathbf{u}' = \mathbf{a}' \otimes \binom{\alpha\gamma}{\alpha\delta+\beta\gamma} + \mathbf{a}'' \otimes \binom{0}{\alpha\gamma} \in \bigotimes_{j=1}^{\delta} \mathbb{C}^2,$$
$$\mathbf{u}'' = \mathbf{a}' \otimes \binom{\beta\delta}{0} + \mathbf{a}'' \otimes \binom{\alpha\delta+\beta\gamma}{\beta\delta} \in \bigotimes_{j=1}^{\delta} \mathbb{C}^2.$$

To obtain $\mathbf{a}', \mathbf{a}'' \in \bigotimes_{j=1}^{\delta-1} \mathbb{C}^2$ from $\mathbf{u}', \mathbf{u}'' \in \bigotimes_{j=1}^{\delta} \mathbb{C}^2$, we introduce

$$\left(\varphi'_\delta(\mathbf{v})\right)_{i_1 i_2 \ldots i_{\delta-1}} := \mathbf{v}_{i_1 i_2 \ldots i_{\delta-1}, 0},$$
$$\left(\varphi''_\delta(\mathbf{v})\right)_{i_1 i_2 \ldots i_{\delta-1}} := \mathbf{v}_{i_1 i_2 \ldots i_{\delta-1}, 1},$$

i.e.,

$$\varphi'_\delta\left(\mathbf{a} \otimes \binom{\alpha}{\beta}\right) = \alpha\mathbf{a}, \qquad \varphi''_\delta\left(\mathbf{a} \otimes \binom{\alpha}{\beta}\right) = \beta\mathbf{a}.$$

4.3 TT Representation of the Convolution Result

THEOREM. Let the tensors $\mathbf{v}, \mathbf{w} \in \bigotimes_{j=1}^{d} \mathbb{C}^2$ be represented by (possibly different) hierarchical formats using the respective subspaces U'_δ and U''_δ, $1 \leq \delta \leq d$, satisfying

$$\begin{array}{lll} U'_1 = \mathbb{C}^2, & U'_\delta \subset U'_{\delta-1} \otimes \mathbb{C}^2, & \mathbf{v} \in U'_d, \\ U''_1 = \mathbb{C}^2, & U''_\delta \subset U''_{\delta-1} \otimes \mathbb{C}^2, & \mathbf{w} \in U''_d. \end{array}$$

The subspaces

$$U_\delta := \mathrm{span}\{\varphi'_{\delta+1}(\mathbf{x} * \mathbf{y}), \varphi''_{\delta+1}(\mathbf{x} * \mathbf{y}) : \mathbf{x} \in U'_\delta,\ \mathbf{y} \in U''_\delta\} \qquad (1 \leq \delta \leq d)$$

satisfy

$$U_1 = \mathbb{C}^2, \quad U_\delta \subset U_{\delta-1} \otimes \mathbb{C}^2, \quad \mathbf{v} * \mathbf{w} \in U_d.$$

The dimension of U_δ can be bounded by

$$\dim(U_\delta) \leq 2 \dim(U'_\delta) \dim(U''_\delta).$$

5 Concluding Remarks

1) Periodic convolution: Compute $u := v \star w \in C^{2n}$ as before and define u^{per} by $u_k^{per} := u_k + u_{k+n}$ $(0 \leq k \leq n-1)$.

2) The general hierarchical format can be used as well.

3) The canonical format or the tensor subspace format ('Tucker') do not seem be work.