Prandtl’s hypersingular integral equation and tensor trains

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Differential equation

Consider a problem of a potential airflow of a rectangular wing by a steady-state stream:

$$\begin{cases} \Delta\Phi = 0 & x \in \mathbb{R}^3 \setminus \Pi, \Pi = [0;1]\times[0;1]\times\{0\} \\ \dfrac{\partial\Phi}{\partial n} = -V_z & x \in [0;1]\times[0;1] \\ \Phi, \nabla\Phi \to 0 & x \to \infty \end{cases}$$

Some properties

- The problem is an exterior Neumann's problem
- The domain is infinite
- Direct approach in numerical solution requires an artificial boundary, which will make the domain finite. Some boundary conditions should be applied on the new boundary
- Let's try to avoid it by reducing 3D differential equation to a 2D integral equation

Prandtl's equation

Let the solution be a double-layer potential
Put it into our equation and get the following integral equation:

$$-\int_0^1\int_0^1 \frac{U(x,y)}{[(x-x_0)^2+(y-y_0)^2]^{3/2}}\,dxdy = F(x_0,y_0) \equiv 4\pi V_z(x_0,y_0,0)$$

Here:

U - surface density of potential

Properties

- Integral does not exist in the classical sense, since it has strong singularity
- So integral should be treated in sense of Hadamard's finite part:

$$I(x_0)=\int_a^b \frac{f(x)}{(x-x_0)^2}dx \equiv \lim_{\varepsilon\to 0}\Big(\int\limits_{[a;b]\setminus(x_0-\varepsilon;x_0+\varepsilon)} \frac{f(x)}{(x-x_0)^2}dx-\frac{2f(x_0)}{\varepsilon}\Big)$$

- It is a first kind integral equation, so we can expect low stability of solution. But, solutions are stable! This is provided by a singularity of integral.
- There is no known analytical solutions of this integral equation for most right sides of equation (e.g. no analytical solution is known for $F(x_0, y_0)\equiv 1$)

Discretization

- Equation could be discretized using discrete vortex method
- Introduce two meshes:

$$\left\{\left(x_i; y_j\right)\right\} \in \Pi,\ (x_{0j}, y_{0j}) \in \Pi,\ 0 \le i, j \le p$$

$$(x_{0j}, y_{0j}) \in (x_{i-1}, x_i) \times (y_{i-1}, y_i)$$

- Suppose that approximate solution is piecewise-constant function

$$u_p(x,y) = \sum_{i=1}^{p} \sum_{j=1}^{p} u_{ij}\phi_{ij} \qquad \phi_{ij} = \begin{cases} 1, & \text{if } \mathrm{x} \in (\mathrm{x}_{\mathrm{i-1}}, \mathrm{x}_{\mathrm{i}}) \times (\mathrm{y}_{\mathrm{j-1}}, y_j) \\ 0, & \text{else} \end{cases}$$

- Substitute solution approximation into the equation

Discretization (cont.)

So we've got a system of linear algebraic equations:

$$\sum_{i=1}^{p}\sum_{j=1}^{p} a_{ij}^{km} u_{ij} = F(x_{0k}, y_{0m}), \quad 1 \le k, m \le p$$

$$a_{ij}^{km} = -\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \frac{dxdy}{[(x-x_0)^2+(y-y_0)^2]^{3/2}} =$$

$$= -\frac{\sqrt{(x_{0k}-x_{i-1})^2+(y_{0m}-y_j)^2}}{(x_{0k}-x_{i-1})(y_{0m}-y_j)} + \frac{\sqrt{(x_{0k}-x_i)^2+(y_{0m}-y_j)^2}}{(x_{0k}-x_i)(y_{0m}-y_j)} -$$

$$-\frac{\sqrt{(x_{0k}-x_i)^2+(y_{0m}-y_{j-1})^2}}{(x_{0k}-x_i)(y_{0m}-y_{j-1})} + \frac{\sqrt{(x_{0k}-x_{i-1})^2+(y_{0m}-y_{j-1})^2}}{(x_{0k}-x_{i-1})(y_{0m}-y_{j-1})}$$

Convergence

The following statements for uniform grids were proved by Lifanov & Poltavsky:

- Integral convergence:

$$\left\| U - U_p \right\|_{L_1[\Pi]} \to 0 \quad p \to \infty$$

- Uniform convergence on any set inside Π:

$$\forall \varepsilon > 0 \quad \delta > 0 \quad \left| U(x_{0k}, y_{0m}) - U_p(x_{0k}, y_{0m}) \right| < A_\delta \left| \ln h \right|^{9/4} h^{1/4 - \varepsilon}$$

Meshes

Consider 2 types of meshes:

1. Uniform mesh
2. Tchebyshev's mesh:

$$x_i = y_i = (1 - \cos(\frac{\pi i}{p}))/2 \quad x_{0i} = y_{0i} = (1 - \cos(\frac{\pi(i-0.5)}{p}))/2$$

Properties of the matrix

- The matrix is dense. Its size is p^2 by p^2
- The matrix is two level Toplitz matrix in case of uniform grid
- The matrix has no prescribed structure in case of Tchebyshev's grid
- The matrix is rather well conditioned:

p/cond	8	16	32
Uniform	5.70	11.02	21.67
Tchebyshev	18.69	74.65	299.48

Approach to solution

- Since the condition number is “adequate”, we can replace the matrix in our problem by its structured approximation
- Using canonical decomposition in this problem was suggested by Oseledets (2005)
- We’re using Tensor Train decomposition

Tensor trains review

- TT decomposition of tensor:

$$B(i_1, i_2, \ldots, i_d) = \sum_{\alpha_1, \ldots a_{d-1}} G_1(i_1, \alpha_1) G_2(\alpha_1, i_2, \alpha_2) \ldots G_{d-1}(\alpha_{d-2}, i_{d-1}, \alpha_{d-1}) G_d(\alpha_{d-1}, i_d)$$

- QTT decomposition of matrix:
  is a TT decomposition of a "virtual tensor"

  $$A\left[i_1, j_1, i_2, j_2, \ldots, i_d, j_d\right]$$

  where

  $$i = \left(i_1, i_2, \ldots, i_d\right),\ j = \left(j_1, j_2, \ldots, j_d\right)$$

- [Q]TT decomposition has many advantages over existing tensor decompositions

Benefits of QTT format

- There exists a cross approximation method for building TT decomposition
- Number of parameters is reduced from p^4 to only $O(4log(p)r^2)$
- Mat-full-vec operation complexity is $O(4r^2p^2log(p))$
- Mat-mat operation complexity is $O(4log(p)r^4)$
- Recompression complexity - $O(4r^3log(p))$

Rank estimation

- The general technique for rank estimation is separation of variables:

$$f(x_1, x_2, \ldots, x_n) \approx \sum f_1(x_1) f_2(x_2) \ldots f_n(x_n)$$

- There are two useful lemmas for rank estimation

Lemma 1 (Oseledets)

For a 2^d x 2^d matrix A its QTT compression ranks r_k satisfy

$$r_k \le trank_{2^k, 2^{d-k}} A$$

trank stands for tensor rank of matrix and equals to minimal r, such that

$$A \approx \sum_{i=1}^{r} U_i \otimes V_i$$

Lemma 2 (Tyrtyshnikov)

Suppose that

$$A[i,j] = f(z'_i, z_j) = f(u,v)$$

$$z_i = (x'_i; y'_i), z_j = (x_i; y_j), u = x'_i - x_j, v = y'_i - y_j$$

If

$$|F(u,v) - F_m(u,v)| < \varepsilon$$

Then

$$A_r = \sum_{i=1}^{r} U_i \otimes V_i, \quad \| A - A_r \|_C < \varepsilon$$

with $r < 2m$

Some recall

In our case elements of matrix are generated by function

$$F(x, y, x_0, y_0) \equiv F(u, v)$$

which consists of four similar additive terms

$$\frac{\sqrt{u^2 + v^2}}{uv}$$

$$F(\ldots) = -\frac{\sqrt{(x_{0k} - x_{i-1})^2 + (y_{0m} - y_j)^2}}{(x_{0k} - x_{i-1})(y_{0m} - y_j)} + \frac{\sqrt{(x_{0k} - x_i)^2 + (y_{0m} - y_j)^2}}{(x_{0k} - x_i)(y_{0m} - y_j)} -$$

$$-\frac{\sqrt{(x_{0k} - x_i)^2 + (y_{0m} - y_{j-1})^2}}{(x_{0k} - x_i)(y_{0m} - y_{j-1})} + \frac{\sqrt{(x_{0k} - x_{i-1})^2 + (y_{0m} - y_{j-1})^2}}{(x_{0k} - x_{i-1})(y_{0m} - y_{j-1})}$$

Obviously,

$$\frac{\sqrt{u^2+v^2}}{uv} = \left(\frac{u}{v}+\frac{v}{u}\right)\frac{1}{\sqrt{u^2+v^2}}$$

The key to success is separation of variables in the square root term.

If square root will be approximated by exponential sums, we'll automatically get separation of its argument variables $u^2 + v^2$

Exponential sums

Result of [Braess, Hackbusch]:

Function

$$\frac{1}{\sqrt{x}}$$

Can be approximated with *r* terms exponential sum on the interval [a; b] with accuracy

$$\varepsilon \le 8\sqrt{r}\left[\omega\left(\frac{1-1/r}{1/\kappa}\right)\right]^{-2r}$$

$$\omega(k) = \exp\left(\frac{\pi K(k)}{K'(k)}\right), \kappa = \frac{a}{b}$$

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, K'(k) = K(\sqrt{1-k^2})$$

Rank estimations

1. Uniform grid:

$$\varepsilon(r) \le 16p\sqrt{4r}\left[\omega\left(\frac{1-4/r}{4p^2}\right)\right]^{-r/8} \le$$

$$16p\sqrt{4r}\exp\left(-\frac{r}{4}\left(\log\left(4\left(\frac{4p^2}{1-4/r}+\frac{1}{2}\right)\right)\right)^{-1}\right)$$

2. Tchebyshev's grid:

$$\varepsilon(r) \le \frac{16}{1-\cos\left(\frac{\pi}{2p}\right)}\sqrt{4r}\left[\omega\left(\frac{4(1-4/r)}{\left(1-\cos\left(\frac{\pi}{2p}\right)\right)^2}\right)\right]^{-r/8} \le$$

$$\frac{16}{1-\cos\left(\frac{\pi}{2p}\right)}\sqrt{4r}\exp\left(-\frac{r}{4}\left(\log\left(4\left(\frac{\left(1-\cos\left(\frac{\pi}{2p}\right)\right)^2}{4(1-4/r)}+\frac{1}{2}\right)\right)\right)^{-1}\right)$$

Theoretical rank values

Uniform grid

p / eps	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
32	187	206	224	242	261
64	220	242	263	284	305
128	256	279	303	327	350
256	292	319	345	371	398
512	331	360	389	418	446
1024	371	402	434	465	497

Theoretical rank values

Tchebyshev's grid

p / eps	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
32	898	974	1050	1126	1201
64	1108	1197	1285	1374	1463
128	1334	1435	1537	1639	1740
256	1576	1690	1805	1919	2034
512	1833	1961	2088	2216	2343
1024	2107	2247	2388	2528	2668

Preconditioning

- Preconditioner is built using Newton's method
- Variants of Newton's method

$$X_k = X_{k-1}(2I - X_{k-1})$$

$$X_k = R[X_{k-1}(2I - X_{k-1})]$$

$$X_k = R[X_{k-1}(2I - X_{k-1})] \quad Y_k = R[Y_{k-1}(2I - X_{k-1})]$$

- All methods have quadratic convergence in practice, but quadratic convergence is not proved for the last one

Numerical experiments

- Krylov's subspaces method is BiCGstab
- The accuracy of method is fixed at a high level
- TT ranks are tuned using known linear combinations of matrix columns

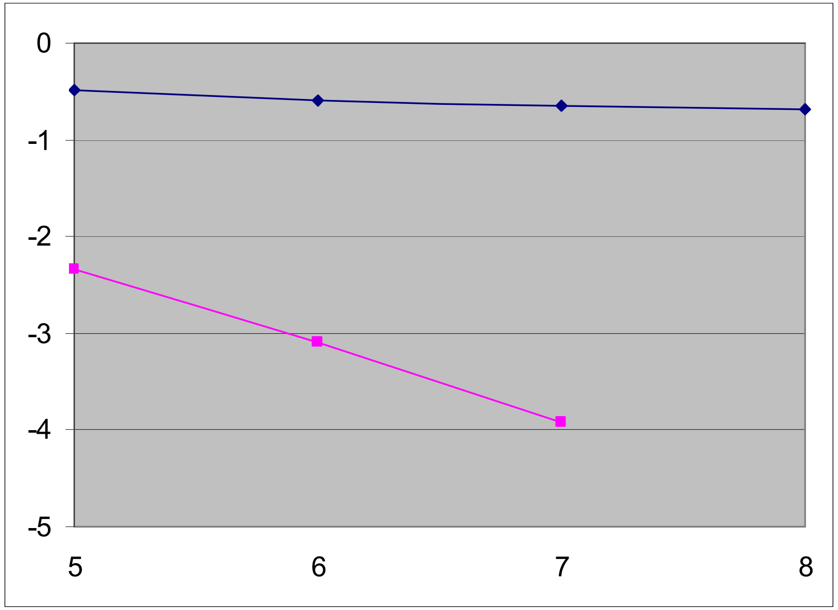
Uniform grid

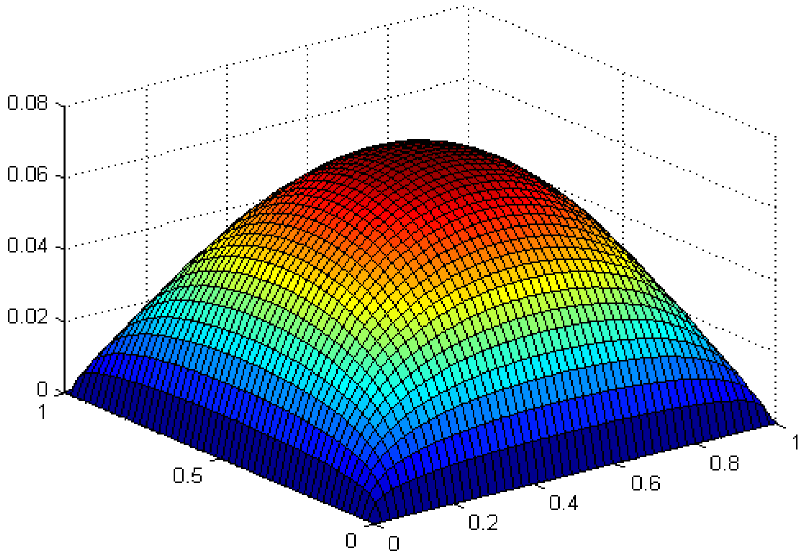
p	64	128	256	512	1024	2048
n	4096	16384	65536	262133	1048576	4194304
Approx. accuracy	10^{-6}	10^{-6}	10^{-6}	10^{-6}	10^{-6}	10^{-6}
Approximation time	21 sec	38 sec	1 min	1.5 min	3 min	6 min
TT rank	25	30	35	40	45	50
Compr. coeff.	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}
Precond. time	2 sec	3 sec	4 sec	5 sec	6 sec	7 sec
Solution time	0.2 sec	2 sec	14 sec	1.5 min	7 min	46 min
Iter. count	3	4	6	7	8	9
Total time	23 sec	43 sec	1 min	3 min	10 min	52 min

Tchebyshev's grid

p	32	64	128	256	512
n	1024	4096	16384	65536	262144
Relative accuracy	10^{-6}	10^{-6}	10^{-5}	10^{-5}	10^{-5}
Approx. time	1 min	4 min	9 min	20 min	42 min
TT rank	53	90	130	170	220
Compr. coeff.	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Precond. time	10 sec	21 sec	35 sec	58 sec	1.5 min
Solution time	0.07 sec	1.13 sec	12 sec	2 min	7 min
Iter. count	3	4	6	9	12
Total time	1.5 min	4.5 min	9.5 min	22 min	51 min

Inner convergence





Other approaches

- Store solution as QTT vector – failed, since solution usually has high rank
- Store solution as QTT matrix (since it is 2 dimensional) – probably, it is a good approach, but it has not been tested yet