Minimal Subspaces

Wolfgang Hackbusch

Max-Planck-Institut für *Mathematik in den Naturwissenschaften*

Inselstr. 22-26, D-04103 Leipzig, Germany
wh@mis.mpg.de

http://www.mis.mpg.de/scicomp/hackbusch_e.html

Leipzig, May 4, 2010

1 Formats Characterised by Subspaces

1) Tensor Subspace Representation (Tucker Format)

For $\mathbf{v} \in \mathbf{V} = V^{(1)} \otimes \ldots \otimes V^{(d)}$ there are subspaces $U^{(j)} \subset V^{(j)}$ of (possibly smaller) dimension $\dim U^{(j)} = r_j$ with

$$\mathbf{u} \in \mathbf{U} = U^{(1)} \otimes U^{(2)} \otimes \ldots \otimes U^{(d)}.$$

Numerical realisation: $U^{(j)} = span\{b_1^{(j)}, \ldots, b_{r_j}^{(j)}\}$ leads to

$$\mathbf{u} = \sum_{1 \leq \ell_1 \leq r_1} \cdots \sum_{1 \leq \ell_d \leq r_d} a_{\ell_1,\ldots,\ell_d} \, b_{\ell_1}^{(1)} \otimes \ldots \otimes b_{\ell_d}^{(d)}.$$

Format: $\mathcal{T}_\mathbf{r} := \left\{ \mathbf{u} \in U^{(1)} \otimes \ldots \otimes U^{(d)} : \dim U^{(j)} = r_j \right\}$

2) Hierarchical Tensor Format

Explanation for the example $d = 4$, $\mathbf{V} = V^{(1)} \otimes V^{(2)} \otimes V^{(3)} \otimes V^{(4)}$

$$
\begin{array}{ccccccc}
\mathbf{V} & = & V^{(1)} \otimes V^{(2)} & \otimes & V^{(3)} \otimes V^{(4)} & & \\
\cup & & \cup \quad \cup & & \cup \quad \cup & & \\
\mathbf{U}' & = & \underbrace{U^{(1)} \otimes U^{(2)}} & \otimes & \underbrace{U^{(3)} \otimes U^{(4)}} & & \dim U^{(j)} = r_j \text{ as in tensor subspace format} \\
 & = & V^{(1+2)} & \otimes & V^{(3+4)} & & \\
\cup & & \cup & & \cup & & \\
\mathbf{U}'' & = & \multicolumn{3}{c}{\underbrace{U^{(1+2)} \quad \otimes \quad U^{(3+4)}}} & & \dim U^{(\alpha)} = r_\alpha \text{ for } \alpha = (1+2)\,,\, (3+4) \\
 & = & & V^{(1-4)} & & & \\
\cup & & & \cup & & & \dim U^{(\alpha)} = r_\alpha \text{ for } \alpha = (1-4) \\
\mathbf{U}''' & = & & U^{(1-4)} & & & \mathbf{v} \in U^{(1-4)}
\end{array}
$$

Format: $\mathcal{H}_\mathbf{r}$

3) TT Format

$$\underbrace{\underbrace{\underbrace{\underset{\supset U_{\{1\}}}{V_1} \otimes V_2}_{\supset U_{\{1,2\}}} \otimes V_3}_{\supset U_{\{1,2,3\}}} \otimes V_4}_{\supset U_{\{1,2,3,4\}}} \otimes V_5 \otimes \ldots$$

$$\rho_j := \dim U_{\{1,\ldots,j\}}$$

Format: $\mathbb{T}_\rho$

Questions:

A) Given $\mathbf{v} \in {}_a\bigotimes_{j=1}^{d} V_j$, what are the subspaces $U_j \subset V_j$ with $\mathbf{v} \in {}_a\bigotimes_{j=1}^{d} U_j$?
What are the minimal subspaces $\longrightarrow U_{j,\mathsf{min}}(\mathbf{v})$.

How does $U_{j,\mathsf{min}}(\mathbf{v})$ depend on $\mathbf{v}$? In particular, $\dim U_{j,\mathsf{min}}(\mathbf{v})$?

B) Best approximation problem: Given $\mathbf{v} \in \bigotimes_{j=1}^{d} V_j$ and some norm $\|\cdot\|$, is there some $\mathbf{u} \in \mathcal{T}_\mathbf{r}$ (or $\in \mathcal{H}_\mathbf{r}$ or $\in \mathbb{T}_\mathbf{r}$) such that

$$\|\mathbf{v} - \mathbf{u}\| = \inf_{\mathbf{w}\in\mathcal{T}_\mathbf{r}} \|\mathbf{v} - \mathbf{w}\|?$$

Case of $\dim V_j = \infty$ included!

2 Minimal Subspaces of a Tensor

General Remarks

REMARK 1.1. For $\mathbf{v} \in {}_a\bigotimes_{j=1}^d V_j$ there are always finite dimensional subspaces $U_j \subset V_j$ satisfying $\mathbf{v} \in {}_a\bigotimes\limits_{j=1}^{d} U_j$.

PROOF: By definition of the algebraic tensor space, $\mathbf{v} \in {}_a\bigotimes_{j=1}^d V_j$ is a *finite* linear combination

$$\mathbf{v} = \sum_{\nu=1}^{n} \bigotimes_{j=1}^{d} v_\nu^{(j)}$$

for some $n \in \mathbb{N}_0$ and $v_\nu^{(j)} \in V_j$. Define

$$U_j := \mathrm{span}\{v_\nu^{(j)} : 1 \leq \nu \leq n\} \qquad \text{for } 1 \leq j \leq d.$$

Then $\mathbf{v} \in \mathbf{U} := {}_a\bigotimes_{j=1}^d U_j$ with $\dim(U_j) \leq n$.

The next, well-known result is formulated for $d = 2$.

LEMMA 1.2. For any tensor $\mathbf{x} \in V \otimes_a W$ there is an $r \in \mathbb{N}_0$ and a representation

$$\mathbf{x} = \sum_{i=1}^{r} v_i \otimes w_i$$

with linearly independent vectors $\{v_i : 1 \leq i \leq r\} \subset V$ and $\{w_i : 1 \leq i \leq r\} \subset W$.

PROOF. Take any representation $\mathbf{x} = \sum_{i=1}^{n} v_i \otimes w_i$. If, e.g., the $\{v_i : 1 \leq i \leq n\}$ are not linearly independent, one v_i can be expressed by the others. Without loss of generality assume $v_n = \sum_{i=1}^{n-1} \alpha_i v_i$. Then

$$v_n \otimes w_n = \left(\sum_{i=1}^{n-1} \alpha_i v_i \right) \otimes w_n = \sum_{i=1}^{n-1} v_i \otimes (\alpha_i w_n)$$

shows that $\mathbf{x}$ possesses a representation with only $n-1$ terms:

$$\mathbf{x} = \left(\sum_{i=1}^{n-1} v_i \otimes w_i \right) + v_n \otimes w_n = \sum_{i=1}^{n-1} v_i \otimes w_i' \qquad \text{with} \quad w_i' := w_i + \alpha_i w_n.$$

Since each reduction step decreases the number of terms by one, this process terminates, i.e., we obtain a representation with linearly independent v_i and w_i.

DEFINITION 1.3. For a tensor $\mathbf{v} \in V_1 \otimes_a V_2$, the *minimal subspaces* are denoted by $U_{1,\mathrm{min}}(\mathbf{v})$ and $U_{2,\mathrm{min}}(\mathbf{v})$ defined by the properties

$$\begin{aligned} \mathbf{v} &\in U_{1,\mathrm{min}}(\mathbf{v}) \otimes_a U_{1,\mathrm{min}}(\mathbf{v}), \\ \mathbf{v} &\in U_1 \otimes_a U_2 \Rightarrow U_{1,\mathrm{min}}(\mathbf{v}) \subset U_1,\ U_{1,\mathrm{min}}(\mathbf{v}) \subset U_2. \end{aligned}$$

To ensure the existence of minimal subspaces U_1, U_2 with $\mathbf{v} \in U_1 \otimes U_2$, we need a *lattice structure*, which is subject of the next lemma.

Then:
Consider the family $\{\left(U_{1,\alpha}, U_{2,\alpha}\right) : \alpha \in \mathcal{F}\}$ of all subspaces such that $\mathbf{v} \in U_{1,\alpha} \otimes_a U_{2,\alpha}$ for all $\alpha \in \mathcal{F}$ and set

$$U_{j,\mathrm{min}}(\mathbf{v}) := \bigcap_{\alpha \in \mathcal{F}} U_{j,\alpha}.$$

LEMMA 1.4. $\mathbf{v} \in X_1 \otimes_a X_2$ and $\mathbf{v} \in Y_1 \otimes_a Y_2 \Rightarrow \mathbf{v} \in (X_1 \cap Y_1) \otimes_a (X_2 \cap Y_2)$.

PROOF. By assumption, $\mathbf{v}$ has the two representations

$$\mathbf{v} = \sum_{\nu=1}^{n_x} x_\nu^{(1)} \otimes x_\nu^{(2)} = \sum_{\nu=1}^{n_y} y_\nu^{(1)} \otimes y_\nu^{(2)} \qquad \text{with } x_\nu^{(i)} \in X_i,\ y_\nu^{(i)} \in Y_i.$$

Thanks to Lemma 1.2, $\{x_\nu^{(1)}\}$, $\{x_\nu^{(2)}\}$, $\{y_\nu^{(1)}\}$, $\{y_\nu^{(2)}\}$ may be assumed to be linearly independent.

Dual basis $\xi_\mu^{(2)} \in X_2$ of $\{x_\nu^{(2)}\}$: $\xi_\mu^{(2)}(x_\nu^{(2)}) = \delta_{\nu\mu}$.

Application of $id \otimes \xi_\mu^{(2)}$ to the first representation: $(id \otimes \xi_\mu^{(2)})(\mathbf{v}) = x_\mu^{(1)}$,

application to second representation: $\sum_{\nu=1}^{n_y} \xi_\mu^{(2)}(y_\nu^{(2)})\, y_\nu^{(1)}$

$\Rightarrow x_\mu^{(1)} = \sum_{\nu=1}^{n_y} \xi_\mu^{(2)}(y_\nu^{(2)})\, y_\nu^{(1)} \Rightarrow x_\nu^{(1)} \in Y_1$.

Dual basis $\xi_\mu^{(1)}$ of $\{x_\nu^{(1)}\}$ and application of $\xi_\mu^{(1)} \otimes id$ to $\mathbf{v}$ proves $x_\nu^{(2)} \in Y_2$.

Hence $x_\nu^{(i)} \in X_i \cap Y_i$ is shown, i.e., $\mathbf{v} \in (X_1 \cap Y_1) \otimes_a (X_2 \cap Y_2)$.

LEMMA 1.5. Assume $\mathbf{v} = \sum_{i=1}^{r} v_i \otimes w_i$ with linearly independent $\{v_i : 1 \le i \le r\}$ and $\{w_i : 1 \le i \le r\}$. Then they span the minimal spaces:

$$U_{1,\mathrm{min}}(\mathbf{v}) = \mathrm{span}\,\{v_\nu : 1 \le \nu \le r\} \quad \text{and} \quad U_{2,\mathrm{min}}(\mathbf{v}) = \mathrm{span}\,\{w_\nu : 1 \le \nu \le r\}.$$

In particular, $\dim(U_{1,\mathrm{min}}(\mathbf{v})) = \dim(U_{2,\mathrm{min}}(\mathbf{v})) = r$.

PROPOSITION 1.6. Let $\mathbf{v} \in V_1 \otimes_a V_2$. The minimal subspaces $U_{1,\mathrm{min}}(\mathbf{v})$ and $U_{2,\mathrm{min}}(\mathbf{v})$ are characterised by

$$U_{1,\mathrm{min}}(\mathbf{v}) = \left\{ (id \otimes \lambda)(\mathbf{v}) : \lambda \in V_2' \right\},$$
$$U_{2,\mathrm{min}}(\mathbf{v}) = \left\{ (\lambda \otimes id)(\mathbf{v}) : \lambda \in V_1' \right\}.$$

$$(id \otimes \lambda)\left(\sum_{i=1}^{r} v_i \otimes w_i\right) = \sum_{i=1}^{r} (id \otimes \lambda)(v_i \otimes w_i) = \sum_{i=1}^{r} \lambda(w_i) \cdot v_i.$$

Matrix interpretation ($\mathbf{v} = M$):

$$U_{1,\mathrm{min}}(\mathbf{v}) = range(M), \quad U_{2,\mathrm{min}}(\mathbf{v}) = range(M^{\mathsf{T}}).$$

COROLLARY 1.7 (a) Once $U_{1,\min}(\mathbf{v})$ and $U_{2,\min}(\mathbf{v})$ are given, one may select any basis $\{v_i : 1 \le i \le n\}$ of $U_{1,\min}(\mathbf{v})$ and finds a representation $\mathbf{v} = \sum_{i=1}^{r} v_i \otimes w_i$ with some basis $\{w_i\}$ of $U_{2,\min}(\mathbf{v})$.
Vice versa, one may select a basis $\{w_i : 1 \le i \le n\}$ of $U_{2,\min}(\mathbf{v})$, and obtains $\mathbf{v} = \sum_{i=1}^{r} v_i \otimes w_i$ with some other basis of $U_{1,\min}(\mathbf{v})$.

(b) If we fix a basis $\{w_\nu : 1 \le \nu \le n\}$ of a subspace U_2, there are mappings $\{\Psi_\nu : 1 \le \nu \le n\} \subset L(V_1 \otimes U_2, V_1)$ such that

$$\mathbf{w} = \sum_{\nu=1}^{n} \Psi_\nu(\mathbf{w}) \otimes w_\nu \qquad \text{for all } \mathbf{w} \in V_1 \otimes U_2.$$

PROOF of (b). Let $\mathbf{w} = \sum_{\nu=1}^{n} v_\nu \otimes w_\nu$.
Choose a dual basis $\{\psi_\nu\}$ to w_ν. i.e., $\psi_\nu(w_\mu) = \delta_{\nu\mu}$.
Set $\Psi_\nu := id \otimes \psi_\nu \Rightarrow$

$$\Psi_\nu(\mathbf{w}) = \sum_{i=1}^{n} (id \otimes \psi_\nu)(v_i \otimes w_i) = \sum_{i=1}^{n} v_i \otimes \psi_\nu(w_i) = v_\nu.$$

Definition in the General Case

Now $d \geq 3$. By Remark 1.1 we may assume $\mathbf{v} \in \mathbf{U} := \bigotimes_{j=1}^{d} U_j$ with *finite* dimensional subspaces $U_j \subset V_j$. The lattice structure from Lemma 1.4 generalises to higher order:

LEMMA 1.8. $\left({}_a\bigotimes_{j=1}^{d} X_j\right) \cap \left({}_a\bigotimes_{j=1}^{d} Y_j\right) = {}_a\bigotimes_{j=1}^{d} \left(X_j \cap Y_j\right)$.

PROOF. Start of the induction at $d = 2$ by Lemma 1.4. Assume assertion for $d-1$ and write ${}_a\bigotimes_{j=1}^{d} X_j$ as $X_1 \otimes X_{[1]}$ with $X_{[1]} := {}_a\bigotimes_{j=2}^{d} X_j$. Similarly, use ${}_a\bigotimes_{j=1}^{d} Y_j = Y_1 \otimes Y_{[1]}$. Lemma 1.4 states that $\mathbf{v} \in (X_1 \cap Y_1) \otimes \left(X_{[1]} \cap Y_{[1]}\right)$. By induction hypothesis, $X_{[1]} \cap Y_{[1]} = {}_a\bigotimes_{j=2}^{d} \left(X_j \cap Y_j\right)$ is valid proving the assertion.

The minimal subspaces $U_{j,\min}(\mathbf{v})$ are given by the intersection of all U_j satisfying $\mathbf{v} \in {}_a\bigotimes_{j=1}^{d} U_j$.

For $\alpha = \bigotimes_{k\neq j} \alpha_k \in {}_a\bigotimes_{k\neq j} V_k'$ define $\alpha : \mathbf{V} \to V_j$ by

$$\alpha(\mathbf{v}) := \left(\alpha_1 \otimes \ldots \otimes \alpha_{j-1} \otimes id \otimes \alpha_{j+1} \otimes \ldots \otimes \alpha_d\right)(\mathbf{v}),$$

$$\alpha\left(\bigotimes_{k=1}^{d} v^{(k)}\right) := \left(\prod_{k\neq j} \alpha_k(v^{(k)})\right) \cdot v^{(j)}.$$

LEMMA 1.9. Let $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j$. The two spaces

$$U_j^{I}(\mathbf{v}) := \left\{\alpha(\mathbf{v}) : \ \alpha \in {}_a\bigotimes_{k\neq j} V_k'\right\},$$

$$U_j^{II}(\mathbf{v}) := \left\{\alpha(\mathbf{v}) : \ \alpha \in ({}_a\bigotimes_{k\neq j} V_k)'\right\}$$

coincide.

PROOF. Mappings α are applied to $\mathbf{v} \in \mathbf{U} := \bigotimes_{j=1}^{d} U_j$, i.e., one may replace $\alpha \in {}_a\bigotimes_{k\neq j} V_k'$ by $\alpha \in {}_a\bigotimes_{k\neq j} U_k'$ and $\alpha \in ({}_a\bigotimes_{k\neq j} V_k)'$ by $\alpha \in ({}_a\bigotimes_{k\neq j} U_k)'$. Since $\dim(U_k) < \infty$, ${}_a\bigotimes_{k\neq j} U_k' = ({}_a\bigotimes_{k\neq j} U_k)'$.

Matricisation: $\mathbf{v} \in V_j \otimes V_{[j]}$ with $V_{[j]} := {}_a\bigotimes_{k \neq j} V_k$.
$\Rightarrow \mathbf{v} \in U_{j,\min}(\mathbf{v}) \otimes V_{[j]}$ with $U_{j,\min} = \left\{ (id \otimes \lambda)(\mathbf{v}) : \lambda \in \left(V_{[j]}\right)' \right\} = U_j^{II}(\mathbf{v})$.

THEOREM 1.10. For any $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j$ there exist minimal subspaces $U_{j,\min}(\mathbf{v})$ $(1 \leq j \leq d)$. An algebraic characterisation of $U_{j,\min}(\mathbf{v})$ is

$$\begin{aligned} & U_{j,\min}(\mathbf{v}) \\ = \; & \mathrm{span}\left\{ \left(\alpha_1 \otimes \ldots \otimes \alpha_{j-1} \otimes id \otimes \alpha_{j+1} \otimes \ldots \otimes \alpha_d\right)(\mathbf{v}) : \; \alpha_k \in V_k' \text{ for } k \neq j \right\} \end{aligned}$$

or equivalently

$$U_{j,\min}(\mathbf{v}) = \left\{ \alpha(\mathbf{v}) : \; \alpha \in {}_a\bigotimes_{k \neq j} V_k' \right\}.$$

3 Topological Tensor Space

$\left(V_j, \|\cdot\|_j\right)$ are Banach spaces. The tensor space $\mathbf{V} := {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j$ is now the completion of the algebraic tensor space ${}_a\bigotimes_{j=1}^{d} V_j$ w.r.t. a norm $\|\cdot\|$.
Notations: V_j^* dual spaces with dual norm $\|\cdot\|_j^*$.

DEFINITION 2.1. For $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j$ define $\|\cdot\|_\vee$ by

$$\|\mathbf{v}\|_\vee := \sup\left\{\frac{\left|\left(\varphi^{(1)} \otimes \varphi^{(2)} \otimes \ldots \otimes \varphi^{(d)}\right)(\mathbf{v})\right|}{\prod_{j=1}^{d} \|\varphi^{(j)}\|_j^*} : 0 \neq \varphi^{(j)} \in V_j^*, 1 \leq j \leq d\right\}.$$

(injective norm [Grothdieck 1953]).

THEOREM 2.2. Any norm $\|\cdot\|$ on ${}_a\bigotimes_{j=1}^{d} V_j$, for which

$$\begin{aligned} \bigotimes_{j=1}^{d} &: V_1 \times \ldots \times V_d \to {}_a\bigotimes_{j=1}^{d} V_j \quad \text{and} \\ \bigotimes_{j=1}^{d} &: V_1^* \times \ldots \times V_d^* \to {}_a\bigotimes_{j=1}^{d} V_j^* \end{aligned}$$

are continuous, cannot be weaker than $\|\cdot\|_\vee$, i.e.,

$$\|\cdot\| \gtrsim \|\cdot\|_\vee . \qquad \text{(norm)}$$

LEMMA 2.3. For fixed $j \in \{1, \ldots, d\}$, $\varphi = \bigotimes_{k \neq j} v_k^* \in {}_a\bigotimes_{k \neq j} V_k^*$ maps $\left({}_a\bigotimes_{k=1}^d V_k \, , \|\cdot\|_\vee \right)$ into V_j via

$$\varphi\left(\bigotimes_{k=1}^d v^{(k)}\right) := \left(\prod_{k \in \{1,\ldots,d\} \setminus \{j\}} v_k^*(v^{(k)})\right) \cdot v^{(j)}.$$

The mapping φ is denoted by $\varphi = v_1^* \otimes \ldots \otimes v_{j-1}^* \otimes id \otimes v_{j+1}^* \otimes \ldots \otimes v_d^*$. Then φ is continuous, i.e., $\varphi \in \mathcal{L}\left({}_\vee\bigotimes_{k=1}^d V_k, V_j\right)$. Its norm is

$$\|\varphi\|_{V_j \leftarrow {}_\vee\bigotimes_{k=1}^d V_k} = \prod_{k \in \{1,\ldots,d\} \setminus \{j\}} \|v_k^*\|_k^* \, .$$

PROOF. Let $v_j^* \in V_j^*$ with $\|v_j^*\|_j^* = 1$ and note that the composition $v_j^* \circ \varphi$ equals $\mathbf{v}^* := \bigotimes_{k=1}^d v_k^*$. Hence,

$$\|\varphi(\mathbf{v})\|_j = \max_{\|v_j^*\|_j^*=1} \left|v_j^*(\varphi(\mathbf{v}))\right| = \max_{\|v_j^*\|_j^*=1} \left|(v_j^* \circ \varphi)(\mathbf{v})\right| = \max_{\|v_j^*\|_j^*=1} \left|\left(\bigotimes_{k=1}^d v_k^*\right)(\mathbf{v})\right|$$

and

$$\sup_{\|\mathbf{v}\|_\vee=1} \left|v_j^*(\varphi(\mathbf{v}))\right| = \sup_{\|\mathbf{v}\|_\vee=1} \left|\left(\bigotimes_{k=1}^d v_k^*\right)(\mathbf{v})\right| = \left\|\bigotimes_{k=1}^d v_k^*\right\|_\vee^* = \prod_{k=1}^d \|v_k^*\|^* = \prod_{k \neq j} \|v_k^*\|_k^* \, .$$

LEMMA 2.4. For $\mathbf{v} \in \mathbf{V} = {}_a\bigotimes_{j=1}^{d} V_j$ define

$$U_j^{III}(\mathbf{v}) := \left\{ \alpha(\mathbf{v}) : \ \alpha \in {}_a\bigotimes_{k \neq j} V_k^* \right\},$$
$$U_j^{IV}(\mathbf{v}) := \left\{ \alpha(\mathbf{v}) : \ \alpha \in ({}_{\|\cdot\|}\bigotimes_{k \neq j} V_k)^* \right\}.$$

Then $U_j^{III}(\mathbf{v}) = U_j^{IV}(\mathbf{v}) = U_{j,\mathsf{min}}(\mathbf{v})$.

PROOF. Hahn-Banach

Sequences of Minimal Subspaces

$\mathbf{V} := {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j$ Banach tensor space with $\|\cdot\|$ satisfying (norm): $\|\cdot\| \gtrsim \|\cdot\|_{\vee}$.

The following definition $U_{j,\min}(\mathbf{v})$ can also be applied to $\mathbf{v} \in {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j \setminus {}_a\bigotimes_{j=1}^{d} V_j$:

$U_{j,\min}(\mathbf{v})$
$= closure\left(\mathrm{span}\left\{\left(\varphi_1 \otimes \ldots \otimes \varphi_{j-1} \otimes id \otimes \varphi_{j+1} \otimes \ldots \otimes \varphi_d\right)(\mathbf{v}) : \varphi_\nu \in V_\nu^* \text{ for } \nu \neq j\right\}\right)$
$= closure\left\{\varphi(\mathbf{v}) :\ \varphi \in {}_a\bigotimes_{k\neq j} V_k^*\right\} = closure\left\{\varphi(\mathbf{v}) :\ \varphi \in \left({}_{\|\cdot\|}\bigotimes_{k\neq j} V_k\right)^*\right\}.$

By LEMMA 2.3, the involved mappings $\varphi : \mathbf{V} \to V_j$ are continuous.

Note that

▲ for $\mathbf{v} \in {}_a\bigotimes_{j=1}^{d} V_j$ the subspace $U_{j,\min}(\mathbf{v})$ is defined by the minimality condition. Thm 1.10 shows $U_{j,\min}(\mathbf{v}) = \mathrm{span}\left\{\left(\bigotimes_{k\neq j}\alpha_k\right)(\mathbf{v}) : \alpha_k \in V_k'\right\}$, which for these $\mathbf{v}$ coincides with the definition above.

▲ for $\mathbf{v} \in {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j \setminus {}_a\bigotimes_{j=1}^{d} V_j$ the equation above is a definition. Whether $\mathbf{v} \in {}_{\|\cdot\|}\bigotimes_{j=1}^{d} U_{j,\min}(\mathbf{v})$ with minimal $U_{j,\min}(\mathbf{v})$ holds, is not yet stated.

A sequence $x_n \in X$ is *weakly convergent*, if $\lim_{n\to\infty} \varphi(x_n)$ exists for all $\varphi \in X^*$. $(x_n)_{n\in\mathbb{N}}$ *converges weakly to* $x \in X$, if $\lim \varphi(x_n) = \varphi(x)$ for all $\varphi \in X^*$. Notation: $x_n \rightharpoonup x$.

LEMMA 2.5. Assume (norm). Let $\varphi_{[j]} \in {}_a\bigotimes_{k\neq j} V_k^*$ and $\mathbf{v}_n, \mathbf{v} \in \mathbf{V}$ with $\mathbf{v}_n \rightharpoonup \mathbf{v}$. (a) Then $\varphi_{[j]}(\mathbf{v}_n) \rightharpoonup \varphi_{[j]}(\mathbf{v})$ in V_j.
(b) The estimate

$$\|\varphi_{[j]}(\mathbf{v} - \mathbf{v}_n)\|_{V_j} \leq C \|\mathbf{v} - \mathbf{v}_n\|$$

holds for elementary tensors $\varphi_{[j]} \in {}_a\bigotimes_{k\neq j} V_k^*$ with $\left\|v_k^*\right\|^* = 1$, where C is the norm constant involved in (Norm).

PROOF. 1) Let $\varphi_{[j]} = \bigotimes_{k\neq j} v_k^*$ $(v_k^* \in V_k^*)$. To prove $\varphi_{[j]}(\mathbf{v}_n) \rightharpoonup \varphi_{[j]}(\mathbf{v})$ show for all $\varphi_j \in V_j^*$ that $\varphi_j(\varphi_{[j]}(\mathbf{v}_n)) \to \varphi_j(\varphi_{[j]}(\mathbf{v}))$. The composition $\varphi_j \circ \varphi_{[j]} = \bigotimes_{k=1}^d v_k^*$ belongs to ${}_\vee\bigotimes_{k=1}^d V_k^*$ and because of (norm) also to $\mathbf{V}^* = ({}_{\|\cdot\|}\bigotimes_{k=1}^d V_k)^*$. Hence $\mathbf{v}_n \rightharpoonup \mathbf{v}$ implies $\varphi_j(\varphi_{[j]}(\mathbf{v}_n)) \to \varphi_j(\varphi_{[j]}(\mathbf{v}))$ and proves Part (a) for an elementary tensor $\varphi_{[j]}$. The result extends immediately to finite linear combinations $\varphi_{[j]} \in {}_a\bigotimes_{k\neq j} V_k^*$.
2) LEMMA 2.3 proves $\|\varphi_{[j]}(\mathbf{v} - \mathbf{v}_n)\|_{V_j} \leq \left(\prod_{k\neq j} \left\|v_k^*\right\|^*\right) \|\mathbf{v} - \mathbf{v}_n\|_\vee$. Estimate $\|\cdot\|_\vee \leq C \|\cdot\|$ shows the inequality of the Lemma.

MAIN THEOREM 2.6. Assume (norm). For $\mathbf{v}_n \in {}_a\bigotimes_{j=1}^d V_j$ assume $\mathbf{v}_n \rightharpoonup \mathbf{v} \in {}_{\|\cdot\|}\bigotimes_{j=1}^d V_j$. Then

$$\dim U_{j,\min}(\mathbf{v}) \leq \liminf_{n\to\infty} \dim U_{j,\min}(\mathbf{v}_n) \qquad \text{for all } 1 \leq j \leq d.$$

Lemma needed for the proof:

LEMMA 2.7. Assume $N \in \mathbb{N}$ and $x_n^{(i)} \rightharpoonup x_\infty^{(i)}$ for $1 \leq i \leq N$ with linearly independent $x_\infty^{(i)} \in X$. Then there is an n_0 such that for all $n \geq n_0$ the N-tuples $(x_n^{(i)} : 1 \leq i \leq N)$ are linearly independent.

PROOF (L. 2.7). There are functionals $\varphi^{(j)} \in X^*$ $(1 \leq j \leq N)$ with $\varphi^{(j)}(x_\infty^{(i)}) = \delta_{ij}$. Set

$$\Delta_n := \det\left((\varphi^{(j)}(x_n^{(i)}))_{i,j=1}^N\right).$$

$x_n^{(i)} \rightharpoonup x_\infty^{(i)}$ implies $\varphi^{(j)}(x_n^{(i)}) \to \varphi^{(j)}(x_\infty^{(i)})$. Continuity of $\det(\cdot)$ proves $\Delta_n \to \Delta_\infty := \det(\left(\delta_{ij}\right)_{i,j=1}^N) = 1$. Hence, there is an n_0 such that $\Delta_n > 0$ for all $n \geq n_0$, but $\Delta_n > 0$ proves linear independence of $\{x_n^{(i)} : 1 \leq i \leq N\}$.

PROOF (Thm 2.6). Choose a subsequence such that $\dim U_{j,\min}(\mathbf{v}_n)$ is weakly increasing. If $\dim U_{j,\min}(\mathbf{v}_n) \to \infty$ holds, nothing is to be proved. Therefore, assume that

$$\lim \dim U_{j,\min}(\mathbf{v}_n) = N < \infty.$$

Indirect proof: assume that $\dim U_{j,\min}(\mathbf{v}) > N$. Since $\left\{\varphi(\mathbf{v}) : \ \varphi \in {}_a\bigotimes_{k\neq j} V_k^*\right\}$ is dense in $U_{j,\min}(\mathbf{v}_n)$, there are $N+1$ linearly independent vectors

$$b^{(i)} = \varphi_{[j]}^{(i)}(\mathbf{v}) \qquad \text{with } \varphi_{[j]}^{(i)} \in {}_a\bigotimes_{k\neq j} V_k^* \quad \text{for } 1 \leq i \leq N+1.$$

By LEMMA 2.5, weak convergence

$$b_n^{(i)} := \varphi_{[j]}^{(i)}(\mathbf{v}_n) \rightharpoonup b^{(i)}$$

holds. By LEMMA 2.7, for large enough n also $(b_n^{(i)} : 1 \leq i \leq N+1)$ is linearly independent. Because of $b_n^{(i)} = \varphi_{[j]}^{(i)}(\mathbf{v}_n) \in U_{j,\min}(\mathbf{v}_n)$, this contradicts $\dim U_{j,\min}(\mathbf{v}_n) \leq N$.

Application to Tensor Formats

The formats $\mathcal{T}_{\mathbf{r}}$ (Tucker), $\mathcal{H}_{\mathbf{r}}$ (hierarchical), $\mathbb{T}_{\mathbf{r}}$ (TT) are essentially determined by the dimension r_α of the involved subspaces $U_\alpha \subset \bigotimes_{k\in\alpha} V_k$.

DEFINITION 2.9. $M \subset X$ is called *weakly closed*, if $x_n \in M$ and $x_n \rightharpoonup x \in X$ imply $x \in M$.

Note that 'weakly closed' is stronger than 'closed', i.e.,
M weakly closed $\Rightarrow$ M closed.

THEOREM 2.8. Under assumption (norm), the sets $\mathcal{T}_{\mathbf{r}}$, $\mathcal{H}_{\mathbf{r}}$, $\mathbb{T}_{\mathbf{r}}$ are weakly closed.

PROOF. Let $\mathbf{v}_n \in \mathcal{T}_{\mathbf{r}}$, i.e., there are subspaces $U_{j,n}$ with $\mathbf{v}_n \in \bigotimes_{j=1}^{d} U_{j,n}$ and $\dim U_{j,n} \le r_j$. Note that $U_{j,\min}(\mathbf{v}_n) \subset U_{j,n}$ with $\dim U_{j,\min}(\mathbf{v}_n) \le r_j$.

If $\mathbf{v}_n \rightharpoonup \mathbf{v}$, then $\dim U_{j,\min}(\mathbf{v}) \le r_j$ and $\mathbf{v} \in \mathcal{T}_{\mathbf{r}}$ (cf. THEOREM 3.1).

Same argument for $\mathcal{H}_{\mathbf{r}}$, $\mathbb{T}_{\mathbf{r}}$.

Application to Best Approximation

THEOREM 2.10. Let $(X, \|\cdot\|)$ be a reflexive Banach space with a weakly closed subset $\emptyset \neq M \subset X$. Then for any $x \in X$ there exists a best approximation $v \in M$ with

$$\|x - v\| = \inf\{\|x - w\| : w \in M\}.$$

LEMMA 2.11. If $x_n \rightharpoonup x$, then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.

LEMMA 2.12. If X is a reflexive Banach space, any bounded sequence $x_n \in X$ has a subsequence x_{n_i} converging weakly to some $x \in X$.

PROOF of Thm 2.10. Choose $w_n \in M$ with $\|x - w_n\| \to \inf\{\|x - w\| : w \in M\}$. Since $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence in X, LEMMA 2.12 ensures the existence of a subsequence $w_{n_i} \rightharpoonup v \in X$. v belongs to M because $w_{n_i} \in M$ and M is weakly closed. Since also $x - w_{n_i} \rightharpoonup x - v$, LEMMA 2.11 shows $\|x - v\| \leq \liminf \|x - w_{n_i}\| \leq \inf\{\|x - w\| : w \in M\}$.

Conclusion for $M \in \{\mathcal{T}_\mathbf{r}, \mathcal{H}_\mathbf{r}, \mathbb{T}_\mathbf{r}\}$:

COROLLARY 2.13: If (norm), best approximations in the formats $\mathcal{T}_\mathbf{r}$, $\mathcal{H}_\mathbf{r}$, $\mathbb{T}_\mathbf{r}$ exist.

4 Minimal Subspaces in Topological Tensor Spaces

Where is the Problem?

For a general $\mathbf{v} \in {}_{\|\cdot\|}\bigotimes_{j=1}^{d} V_j$ the subspace $U_{j,\min}(\mathbf{v})$ is defined. Set

$$\mathbf{U}(\mathbf{v}) := {}_{\|\cdot\|}\bigotimes_{j=1}^{d} U_{j,\min}(\mathbf{v}) \,.$$

One expects the statement

$$\mathbf{v} \in \mathbf{U}(\mathbf{v}).$$

We shall prove this statement in three different situations.

A) $\dim U_{j,\min}(\mathbf{v}) < \infty$,
B) Hilbert space setting,
C) convergence $\mathbf{v}_n \to \mathbf{v}$ $(tensorrank(\mathbf{v}_n) \leq n)$ fast enough.

Note that the definition of minimal subspaces via the lattice property is not obvious:

$$\mathbf{v} \in closure\,(X_1 \otimes_a X_2) \cap closure\,(Y_1 \otimes_a Y_2) \not\Rightarrow \mathbf{v} \in closure\,((X_1 \cap Y_1) \otimes_a (X_2 \cap Y_2))$$

THEOREM 3.1. Assume (norm), $\mathbf{v} \in \mathbf{V} = {}_{\|\cdot\|} \bigotimes_{j=1}^{d} V_j$ with $\dim(U_{j,\min}(\mathbf{v})) < \infty$. Then $\mathbf{v}$ belongs to the *algebraic* tensor space $\mathbf{U} = {}_a \bigotimes_{j=1}^{d} U_{j,\min}(\mathbf{v})$.

LEMMA of Auerbach 3.2. For any n-dimensional subspace of a Banach space X, there exists a basis $\{x_\nu : 1 \le \nu \le n\}$ and a corresponding dual basis $\{\varphi_\nu : 1 \le \nu \le n\}$ such that $\|x_\nu\| = \|\varphi_\nu\|^* = 1$ $(1 \le \nu \le n)$.

THEOREM 3.3. $Y \subset X$ be subspace of a Banach space X with $\dim(Y) \le n$. Then there exists a projection $\Phi \in \mathcal{L}(X, X)$ onto Y such that

$$\|\Phi\|_{X \leftarrow X} \le n^{1/2}.$$

The bound is sharp for general Banach spaces, but can be improved to $\|\Phi\|_{X \leftarrow X} \le n^{1/2 - 1/p}$ for $X = L^p$.

THEOREM 3.4. Let $\|\cdot\|$ be a crossnorm. Assume that $\mathbf{v} \in \mathbf{V} = {}_{\|\cdot\|} \bigotimes_{j=1}^{d} V_j$ is the limit of $\mathbf{v}_n = \sum_{i=1}^{n} \bigotimes_{j=1}^{d} v_{i,n}^{(j)}$ with the speed

$$\|\mathbf{v}_n - \mathbf{v}\| \le o(n^{-3/2}).$$

Then $\mathbf{v} \in \mathbf{U} = {}_{\|\cdot\|} \bigotimes_{j=1}^{d} U_{j,\min}(\mathbf{v})$, i.e., $\mathbf{v} = \lim \mathbf{u}_n$ with $\mathbf{u}_n \in {}_a \bigotimes_{j=1}^{d} U_{j,\min}(\mathbf{v})$.

PROOF. Consider $\mathbf{V} = V_1 \otimes V_{[1]}$. $\mathbf{v}_n \in \mathbf{V}$ has a representation in $U_{1,\min}(\mathbf{v}_n) \otimes U_{[1],\min}(\mathbf{v}_n)$ with $r := \dim(U_{1,\min}(\mathbf{v}_n)) = \dim U_{[1],\min}(\mathbf{v}_n) \leq n$. Renaming n by r, we obtain the representation $\mathbf{v}_n = \sum_{i=1}^n v_i^{(1)} \otimes v_i^{[1]}$. According to COROLLARY 1.7, we can fix a basis $\{v_i^{[1]}\}$ and recover $\mathbf{v}_n = \sum_{i=1}^n \psi_i(\mathbf{v}_n) \otimes v_i^{[1]}$ from the dual basis $\{\psi_i\}$. Define

$$\mathbf{u}_n^I := \sum_{i=1}^n \psi_i(\mathbf{v}) \otimes v_i^{[1]} \in U_{1,\min}(\mathbf{v}) \otimes_a V_{[1]}.$$

Due to COROLLARY 1.7, the choice of the basis $\{v_i^{[1]}\}$ is arbitrary. Choosing the basis according to Auerbach's LEMMA 3.2, we can estimate

$$\begin{aligned} \|\mathbf{u}_n^I - \mathbf{v}_n\| &= \left\| \sum_{i=1}^n (\psi_i(\mathbf{v}) - \psi_i(\mathbf{v}_n)) \otimes v_i^{[1]} \right\| \leq \sum_{i=1}^n \|\psi_i(\mathbf{v} - \mathbf{v}_n)\| \|v_i^{[1]}\| \\ &\leq \left(\sum_{i=1}^n \|\psi_i\|^* \|v_i^{[1]}\| \right) \|\mathbf{v} - \mathbf{v}_n\| \underset{\text{Auerbach}}{\leq} n \|\mathbf{v} - \mathbf{v}_n\|. \end{aligned}$$

Analogously, we can choose the basis $(v_i^{(1)})$ according to Auerbach's LEMMA 3.2 and define $\mathbf{u}_n^{II} := \sum_{i=1}^n v_i^{(1)} \otimes \psi_i(\mathbf{v}) \in V_1 \otimes_a U_{[1],\min}(\mathbf{v})$ with

$$\|\mathbf{u}_n^{II} - \mathbf{v}_n\| \leq n \|\mathbf{v} - \mathbf{v}_n\|.$$

$$\|\mathbf{u}_n^I - \mathbf{v}_n\| \le n\|\mathbf{v} - \mathbf{v}_n\| \text{ and } \|\mathbf{u}_n^{II} - \mathbf{v}_n\| \le n\|\mathbf{v} - \mathbf{v}_n\|.$$

We choose the projection Φ onto the subspace $U_{[1],\min}(\mathbf{v})$ of dimension n from THEOREM 3.3 and define

$$\mathbf{u}_n := (id \otimes \Phi)\, \mathbf{u}_n^I \in U_{1,\min}(\mathbf{v}) \otimes_a U_{[1],\min}(\mathbf{v}) \subset {}_a \bigotimes_{j=1}^{d} U_{j,\min}(\mathbf{v})\,.$$

THEOREM 3.3 implies the estimates

$$\|\,(id \otimes \Phi)\, \mathbf{v}_n - \mathbf{u}_n^{II}\| = \|\,(id \otimes \Phi) \left(\mathbf{v}_n - \mathbf{u}_n^{II}\right)\| \le n^{1/2}\|\mathbf{v}_n - \mathbf{u}_n^{II}\| \le n^{3/2}\|\mathbf{v} - \mathbf{v}_n\|,$$
$$\|\mathbf{u}_n - (id \otimes \Phi)\, \mathbf{v}_n\| = \|\,(id \otimes \Phi) \left(\mathbf{u}_n^I - \mathbf{v}_n\right)\| \le n^{1/2}\|\mathbf{u}_n^I - \mathbf{v}_n\| \le n^{3/2}\|\mathbf{v} - \mathbf{v}_n\|.$$

Altogether we get the estimate

$$\begin{aligned}\|\mathbf{u}_n - \mathbf{v}\| &= \|\,(\mathbf{u}_n - (id \otimes \Phi)\, \mathbf{v}_n) + \left((id \otimes \Phi)\, \mathbf{v}_n - \mathbf{u}_n^{II}\right) + \left(\mathbf{u}_n^{II} - \mathbf{v}_n\right) + (\mathbf{v}_n - \mathbf{v})\,\| \\ &\le \left(2n^{3/2} + n + 1\right) \|\mathbf{v} - \mathbf{v}_n\|.\end{aligned}$$

The assumption $\|\mathbf{v} - \mathbf{v}_n\| \le o(n^{-3/2})$ implies $\|\mathbf{u}_n - \mathbf{v}\| \to 0$.

THEOREM 3.5. V_j and $\mathbf{V}$ Hilbert spaces. Then for all $\mathbf{v} \in \mathbf{V}$, $\mathbf{v} \in {}_{\|\cdot\|}\bigotimes_{j=1}^{d} U_{j,\mathsf{min}}(\mathbf{v})$.

PROOF. The orthogonal decomposition $V_j = U_{j,\mathsf{min}}(\mathbf{v}) + U_{j,\mathsf{min}}^{\perp}(\mathbf{v})$ defines the orthogonal projection P_j onto $U_{j,\mathsf{min}}(\mathbf{v})$. Then $\mathbf{P} := \bigotimes_{j=1}^{d} P_j$ is the projection of $\mathbf{V}$ onto $\mathbf{U} :=_{\|\cdot\|} \bigotimes_{j=1}^{d} U_{j,\mathsf{min}}(\mathbf{v})$. It is easy to verify that $w_j \in U_{j,\mathsf{min}}^{\perp}(\mathbf{v})$ leads to $\left(id \otimes \ldots \otimes w_j \otimes \ldots \otimes id\right)(\mathbf{v}) = 0$ and to $\left(id \otimes \ldots \otimes P_j \otimes \ldots \otimes id\right)(\mathbf{v}) = \mathbf{v}$. Since $\prod_{j=1}^{d} \left(id \otimes \ldots \otimes P_j \otimes \ldots \otimes id\right) = \mathbf{P}$, the equation $\mathbf{P}\mathbf{v} = \mathbf{v}$ shows $\mathbf{v} \in \mathbf{U}$.

To make the last conclusion more explicit, consider a sequence

$$\mathbf{v}_n = \sum_{i=1}^{m(n)} \bigotimes_{j=1}^{d} v_{i,n}^{(j)} \in {}_a\bigotimes_{j=1}^{d} V_j \quad \text{with} \quad v_{i,n}^{(j)} \in V_j \quad \text{and} \quad \|\mathbf{v} - \mathbf{v}_n\| \to 0.$$

Set $\mathbf{u}_n := \mathbf{P}\mathbf{v}_n = \sum_{i=1}^{m(n)} \bigotimes_{j=1}^{d} P_j v_{i,n}^{(j)} \in {}_a\bigotimes_{j=1}^{d} U_{j,\mathsf{min}}(\mathbf{v})$. Since $\mathbf{P}\mathbf{v} = \mathbf{v}$, one obtains the estimate $\|\mathbf{v} - \mathbf{u}_n\| = \|\mathbf{v} - \mathbf{P}\mathbf{v}_n\| = \|\mathbf{P}\left(\mathbf{v} - \mathbf{v}_n\right)\| \leq \|\mathbf{v} - \mathbf{v}_n\| \to 0$, i.e., there is a sequence in ${}_a\bigotimes_{j=1}^{d} U_{j,\mathsf{min}}(\mathbf{v})$ which converges to $\mathbf{v}$.

5 Intersection Tensor Spaces

Problem: For spaces like $C^1(I \times J) = C^1(I) \otimes_{\|\cdot\|} C^1(J)$ or $H^1(I \times J) = H^1(I) \otimes_{\|\cdot\|} H^1(J)$, the tensor product of the dual spaces is not continuous and the injective norm is not bounded.
In the following, the Sobolev space H^1 is taken as example.

Algebraic Intersections Spaces

For $I_j \subset \mathbb{R}$ set $\mathbf{I} := I_1 \times \ldots \times I_d$. Two possibilities to define $H^1(\mathbf{I})$:

$$
\begin{aligned}
H^1(\mathbf{I}) &= closure\Big({}_a\bigotimes_{j=1}^{d} H^1(I_j)\Big), \\
H^1(\mathbf{I}) &= closure(H^1(I_1) \otimes_a L^2(I_2) \otimes_a \ldots \otimes_a L^2(I_d)) \\
&\quad \cap closure(L^2(I_1) \otimes_a H^1(I_2) \otimes_a \ldots \otimes_a L^2(I_d)) \cap \ldots
\end{aligned}
$$

Standard definition of the induced scalar product would give $H^1_{\mathsf{mix}}(\mathbf{I})$.

An algebraic tensor $\mathbf{v}$ leads to

$$U_{j,\mathsf{min}}(\mathbf{v}) \subset H^1(I_j), \quad \mathbf{v} \in \mathbf{U} := {}_a\bigotimes_{j=1}^{d} U_{j,\mathsf{min}}(\mathbf{v}) \subset {}_a\bigotimes_{j=1}^{d} H^1(I_j) \subset H^1_{\mathsf{mix}}(\mathbf{I}).$$

Topological Intersections Spaces

$$U_{j,\mathsf{min}}(\mathbf{v}) = closure\left\{\varphi(\mathbf{v}) : \ \varphi \in {}_a\bigotimes\nolimits_{k\neq j} L^2(I_k)\right\}.$$

Condition (norm) becomes

$\|\cdot\|_{\vee(L^2(I_1),\ldots,L^2(I_{j-1}),H^1(I_j),L^2(I_{j+1}),\ldots,L^2(I_d))} \lesssim \|\cdot\|_{H^1(\mathbf{I})}$ for all $1 \leq j \leq d$

(of course satisfied for $H^1(\mathbf{I})$).

Then analogous conclusions.