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Image of Abel–Jacobi map for hyperelliptic genus 3 and 4 curves

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Abstract

The image of Abel–Jacobi embedding of a hyperelliptic Riemann surface of genus less than 5 to its Jacobian is the intersection of several shifted theta divisors with carefully chosen shifts.

Keywords: Riemann surface; Abelian integral; Abel–Jacobi map; (non) special divisor

The necessity for computation of abelian integrals often arises in problems of classical mechanics [11,7], conformal mappings of polygons [4,12,17], solitonic dynamics [1], general relativity [14,8], rational optimization [3,5]—see also [2] and references in the mentioned sources. Function theory on Riemann surfaces allows one to evaluate with computer accuracy such integrals without any quadrature rules. As an example, let us consider Riemann's formula for the

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abelian integral with two simple poles at the points R, Q of the curve $\mathcal{X}$:

$$\eta_{RQ}(P) := \int^P d\eta_{RQ} = \log \frac{\theta[\epsilon, \epsilon'](u(P) - u(R), \Pi)}{\theta[\epsilon, \epsilon'](u(P) - u(Q), \Pi)} + \text{const}, \tag{1}$$

where $\theta[\cdot](\cdot, \cdot)$ is Riemann theta function with some odd integer characteristics $[\epsilon, \epsilon']$. It might seem that the usage of this formula still requires the computation of holomorphic abelian integrals involved in the Abel–Jacobi (AJ) map:

$$u(P) := \int_{P_0}^P du \in \mathbb{C}^g, \quad du := (du_1, du_2, \ldots, du_g)^t, \tag{2}$$

where g is the genus of the curve $\mathcal{X}$ and du_s are suitably normalized abelian differentials of the first kind.

In this note we show how to avoid the evaluation of AJ map: the image of low genus $g < 5$ hyperelliptic curve in its Jacobian is given as the solution of a (slightly overdetermined) set of equations containing theta functions. Moving along the curve embedded in its Jacobian we can compute the abelian integral by an explicit formula (e.g. (1)) and simultaneously compute a projection of the curve to the complex projective line (see e.g. [10,4,15]). In this way we get a parametric representation [4,12] of abelian integrals which may be used either for the evaluation of integral, or for its inversion. Alternative function theoretic approaches for the computation of abelian integrals use (hyperelliptic) σ and $\wp$ functions [8] or Schottky functions [4, Chapter 6].

Computation of theta function still requires the knowledge of period matrix Π which should be retrieved from the known moduli of the curve. The latter problem is very important, however it is beyond the scope of this note, see e.g. [9] for the expression of period matrix in terms of theta constants.

The techniques developed in this note were used e.g. for the computation of a conformal grid transferred from a rectangle—see Fig. 1 and [13]. The author thanks Oleg Grigoriev for the computations and the anonymous referee for numerous valuable references.

1. Introduction

Let us fix the notations. Consider genus g hyperelliptic curve $\mathcal{X}$:

$$w^2 = \prod_{j=1}^{2g+2} (x - x_j),$$

with distinct branch points $x_1, x_2, \ldots, x_{2g+2}$. The curve admits involution $J(x, w) := (x, -w)$ with fixed points $P_s = (x_s, 0)$, $s = 1, \ldots, 2g + 2$. We introduce a symplectic basis in the homologies of $\mathcal{X}$ as shown in Fig. 2. Dual basis of holomorphic differentials satisfies normalization conditions

$$\int_{a_j} du_s := \delta_{js},$$

and generates the period matrix

$$\int_{b_j} du_s =: \Pi_{js}.$$

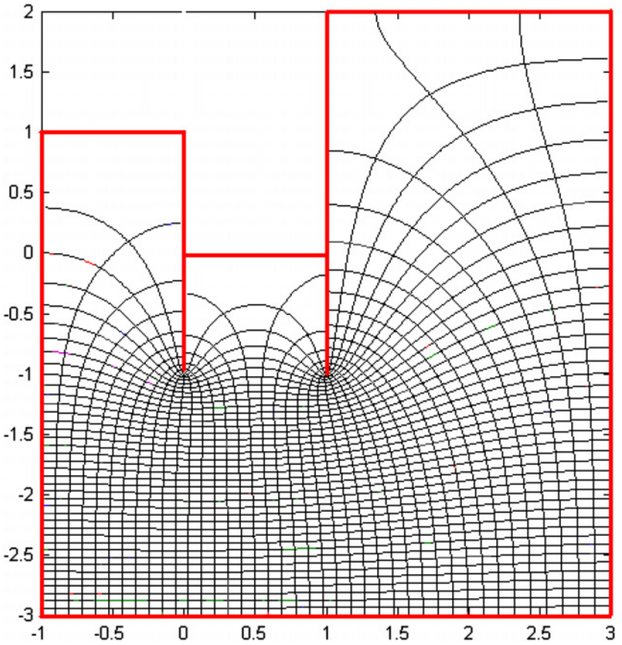

Fig. 1. Conformal grid in a rectangular polygon (computed by O.A. Grigoriev).

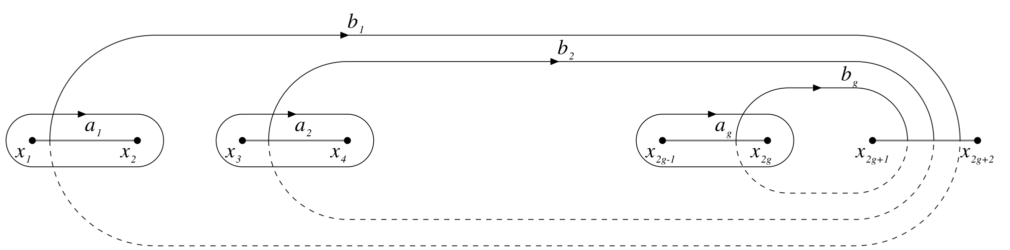

Fig. 2. Symplectic basis in the homologies of curve $\mathcal{X}$.

It is convenient to represent points $u \in \mathbb{C}^g$ as theta characteristics, i.e. couple of real g-vector columns ϵ, ϵ':

$$u = (\epsilon' + \Pi\epsilon)/2. \tag{3}$$

The points of Jacobian

$$Jac(\mathcal{X}) := \mathbb{C}^g / L(\Pi), \quad L(\Pi) := \mathbb{Z}^g + \Pi\mathbb{Z}^g, \tag{4}$$

in this notation correspond to two vectors with real entries modulo 2. Second order points of Jacobian are $2 \times g$ matrices with binary entries. In particular, the images of the Weierstrass points of $\mathcal{X}$ under the AJ map (2) with the initial point $P_0 := P_{2g+2}$ are as follows:

P_s	$u(P_s) \bmod L(\Pi)$	$[\epsilon, \epsilon']^t$
P_1	$\Pi_1/2$	$\begin{bmatrix}100\cdots\\000\cdots\end{bmatrix}$
P_2	$(\Pi_1+E_1)/2$	$\begin{bmatrix}100\cdots\\100\cdots\end{bmatrix}$
P_3	$(\Pi_2+E_1)/2$	$\begin{bmatrix}010\cdots\\100\cdots\end{bmatrix}$
P_4	$(\Pi_2+E_1+E_2)/2$	$\begin{bmatrix}010\cdots\\110\cdots\end{bmatrix}$
P_5	$(\Pi_3+E_1+E_2)/2$	$\begin{bmatrix}0010\cdots\\1100\cdots\end{bmatrix}$
$\vdots$	$\vdots$	$\vdots$
P_{2g-1}	$(\Pi_g+E_1+E_2+\cdots+E_{g-1})/2$	$\begin{bmatrix}0\cdots01\\1\cdots10\end{bmatrix}$
P_{2g}	$(\Pi_g+E_1+E_2+\cdots+E_g)/2$	$\begin{bmatrix}0\cdots01\\11\cdots1\end{bmatrix}$
P_{2g+1}	$(E_1+E_2+\cdots+E_g)/2$	$\begin{bmatrix}00\cdots0\\11\cdots1\end{bmatrix}$
P_{2g+2}	0	$\begin{bmatrix}00\cdots0\\00\cdots0\end{bmatrix}$

Here E_s and Π_s are the columns of the identity matrix and the period matrix respectively.

The following series has very high convergence rate (apply Siegel modular transformations if necessary) and well controlled accuracy [6].

$$\theta(u,\Pi) := \sum_{m\in\mathbb{Z}^g} \exp(2\pi i m^t u + \pi i m^t \Pi m), \quad u\in\mathbb{C}^g;\ \Pi=\Pi^t\in\mathbb{C}^{g\times g};\ \Im\Pi>0. \quad (5)$$

Matrix argument Π of theta function may be omitted if it does not lead to a confusion. This function (5) has the following easily checked quasi-periodicity properties with respect to the lattice $L(\Pi)$:

$$\theta(u+m'+\Pi m;\Pi) = \exp(-i\pi m^t\Pi m - 2i\pi m^t u)\theta(u;\Pi), \quad m,m'\in\mathbb{Z}^g. \quad (6)$$

Theta function may be considered as a multivalued function in the Jacobian or as a section of a certain line bundle. The zero set of theta function – the theta divisor – is well defined in the Jacobian since the exponential factor on the right hand side of (6) does not vanish. The theta divisor is described by so called Riemann vanishing theorems [16,10]. One of the important ingredients in those theorems is a vector of Riemann's constants $\mathcal{K}$ which depends on the choice of homology basis and the initial point in the AJ map. In the above setting the vector of Riemann's constants may be found by a straightforward computation [15] or by some combinatorial argument [10] and corresponds to a characteristic

$$\mathcal{K}(P_0) \sim \begin{bmatrix}\cdots 11111\\ \cdots 10101\end{bmatrix}^t.$$

Theta divisor is described by the following:

Theorem 1 (*Riemann*)**.** *$\theta(e)=0$ iff $e=u(D_{g-1})+\mathcal{K} \bmod L(\Pi)$ for some degree $g-1$ positive divisor D_{g-1} on the curve.*

Here and in what follows we use the standard notions of the function theory on Riemann surfaces [10,1,15]. *D*ivisor is a finite set of points on a surface taken with integer weights: $D=\sum_j m_jP_j,\ m_j\in\mathbb{Z},\ P_j\in\mathcal{X}$. The *d*egree of a divisor is the sum of all weights of participating points: $\deg D := \sum_j m_j$. A *p*ositive divisor is the divisor with positive weights only. The notion of divisor is very natural and useful in e.g. description of zeros/poles of meromorphic functions or differentials on a surface.

*I*ndex of speciality $i(D)$ of a positive divisor D is the dimension of the space of holomorphic differentials vanishing at the points of D (counting multiplicities). Divisors with index $i(D) > g - \deg D$ are called *s*pecial, which means that their points are not in generic positions. Two divisors are *l*inearly equivalent iff their difference is a set of zeros and poles (latter should be taken with negative multiplicities) of a meromorphic function. Riemann–Roch theorem [10] says that class of a positive divisor contains a unique element iff this divisor is not special.

Theorem 1 in particular means that genus 2 curve in its Jacobian is just the solution of one equation $\theta(u + \mathcal{K}) = 0$. Now we consider higher genera.

2. Genus three hyperelliptic curves

Theorem 2. *Let P, Q be two distinct points on the genus 3 hyperelliptic curve $\mathcal{X}$. The solution of the set of two equations*

$$\begin{aligned} &\theta(u - u(P) + \mathcal{K}) = 0, \\ &\theta(u - u(Q) + \mathcal{K}) = 0, \end{aligned} \tag{7}$$

in the Jacobian Jac(X) is the union of $u(\mathcal{X})$– the image of the curve under AJ map – and its shift $u(\mathcal{X}) + u(P + Q)$.

Proof. Let u be a point in the intersection of two shifted theta divisors (7). The representation of the theta divisor from Theorem 1, suggests that there are two degree 2 positive divisors D_2 and D_2' on the curve $\mathcal{X}$ such that

$$u = u(D_2 + P) = u(D_2' + Q) \bmod L(\Pi).$$

We consider two cases:

(1) When two linearly equivalent divisors $D_2 + P \sim D_2' + Q$ are non-special, they coincide and therefore $u \in u(\mathcal{X}) + u(P + Q)$ in the Jacobian of the curve.
(2) When $i(D_2+P) = i(D_2'+Q) > 0$, each divisor contains J-equivalent points which together give zero input to the AJ map. Therefore $u \in u(\mathcal{X})$.

Conversely, consider any point u in the union of $u(\mathcal{X})$ and its shift by $u(P + Q)$. In other words, $u = u(S)$ or $u = u(S + P + Q)$ for some point S of our curve. The first equation in the system (7) is true because the argument of theta function satisfies the condition of Theorem 1 with $D_2 := S + JP$ or $D_2 := S + Q$ respectively. Same argument (with proper choice of D_2) fits for the other equation. ■

We see that two equations are not enough to localize the image of AJ map since the parasitic component $u(\mathcal{X}) + u(P + Q)$ arises. Adding yet another equation of this type, we achieve the goal.

Theorem 3. *Let P, Q, R be three distinct points on the genus 3 hyperelliptic curve $\mathcal{X}$. The solution of the set of three equations*

$$\begin{aligned} \theta(u - u(P) + \mathcal{K}) &= 0, \\ \theta(u - u(Q) + \mathcal{K}) &= 0, \\ \theta(u - u(R) + \mathcal{K}) &= 0 \end{aligned} \tag{8}$$

in the Jacobian Jac$(\mathcal{X})$ is the union of $u(\mathcal{X})$– the image of the curve under AJ map – and just one point $u(P + Q + R)$.

Proof. Taking into account the previous theorem, we have to find the intersection of two different shifts of the same curve $u(\mathcal{X})$: by $u(P+Q)$ and by $u(P+R)$. The point in this intersection has two representations

$$u = u(S+P+Q) = u(S'+P+R)$$

for some points S, S' of the curve. If $i(S+P+Q) = i(S'+P+R) = 0$, then two mentioned divisors coincide: $S = R$, $S' = Q$ and therefore $u = u(P+Q+R)$. Assume that speciality index is positive. Then each of the divisors contains two J-equivalent points whose AJ-images are opposite. Hence, $u \in u(\mathcal{X})$. ■

Remark. There is a temptation to put $P = JQ$ in the system (7) and to get the curve $u(\mathcal{X})$ as a solution of just two equations. However, the only points of the curve with known value of AJ map are its Weierstrass points P_s—see the table above. With this choice of auxiliary points, the systems (8) and the forthcoming system (10) may be rewritten in terms of theta functions with integer characteristics.

3. Genus four hyperelliptic curves

We need an auxiliary statement:

Lemma 1. *Let D_s be a degree s positive divisor on genus 4 hyperelliptic curve and $u(D_3) = u(D_5)$. Then D_5 contains J-equivalent points.*

Proof. Let us detach a point from the larger divisor: $D_5 = D_4 + P$, then D_4 is linear equivalent to $D_3 + JP$. Now if $i(D_4) > 0$, then D_4 contains J-equivalent points. Otherwise D_4 is non-special and it contains JP, hence $D_5 \geq P + JP$. ■

Theorem 4. *Let P_1, P_2; Q_1, Q_2 be four distinct points on the genus 4 hyperelliptic curve $\mathcal{X}$ and $P_1 \neq JP_2$; $Q_1 \neq JQ_2$. The solution of the set of three equations*

$$\begin{aligned} \theta(u+\mathcal{K}) &= 0, \\ \theta(u-u(P_1+P_2)+\mathcal{K}) &= 0, \\ \theta(u-u(Q_1+Q_2)+\mathcal{K}) &= 0, \end{aligned} \tag{9}$$

in the Jacobian is the union of $u(\mathcal{X})$– the image of the curve under AJ map – and its shifts by four vectors $u(P_j + Q_s)$, $j, s = 1, 2$.

Remark. The condition of this theorem implies that $u(P_1 + P_2) \neq u(Q_1 + Q_2) \mod L(\Pi)$. Indeed, if it were not the case, then by Abel's theorem P_1 and P_2 were zeros of degree 2 function from $\mathbb{C}(\mathcal{X})$. This function is essentially unique and therefore $P_1 = JP_2$.

Proof. Let u be a point in the intersection of three shifted theta divisors (9). Due to the representation of the theta divisor, there are three positive divisors D_3, D_3', D_3'', each of degree 3 such that

$$u = u(D_3) = u(D_3' + P_1 + P_2) = u(D_3'' + Q_1 + Q_2) \mod L(\Pi).$$

The divisors $D_5' := D_3' + P_1 + P_2$ and $D_5'' := D_3'' + Q_1 + Q_2$ in two latter equations contain J-equivalent points in accordance with the Lemma 1. We consider two cases:

(1) When $i(D_3) > 1$, then the divisor contains two points which give opposite input to the AJ mapping. Hence, $u \in u(\mathcal{X})$. (2) When $i(D_3) = 1$ (non-special divisor), both divisors D_5' and D_5''

with J-equivalent points thrown away, coincide with D_3. Now D_3 contains at least one point of P_1, P_2 and one point of Q_1, Q_2. Hence, $u \in u(\mathcal{X}) + u(P_j + Q_s)$ for some j and s.

Conversely, for the points u in the shifted AJ-images of the curve $\mathcal{X}$, the arguments of the theta functions in (9) satisfy the condition of the Theorem 1. Say, for the point $u = u(S + P_1 + Q_2)$, $S \in \mathcal{X}$, the divisor $D_3 = S + P_1 + Q_2$ for the first equation; $D_3 = S + JP_2 + Q_2$ for the second equation; $D_3 = S + P_1 + JQ_1$ for the third equation in (9). For $u = u(S)$, the divisor $D_3 = S + P_1 + JP_1$ for the first equation; $D_3 = S + JP_1 + JP_2$ for the second equation; $D_3 = S + JQ_1 + JQ_2$ for the third equation in the system. ■

Adding yet another equation of this type to (9) allows us to localize the image of AJ map for genus 4 hyperelliptic curves. Eight parasitic points however arise in the solution.

Theorem 5. *Let $P_s, Q_s, R_s,\ s = 1, 2$ be six distinct points on the genus 4 hyperelliptic curve $\mathcal{X}$ and the points of each of three pairs P_s, Q_s, R_s are not J-equivalent. The solution of the set of four equations*

$$\begin{aligned} \theta(u + \mathcal{K}) &= 0, \\ \theta(u - u(P_1 + P_2) + \mathcal{K}) &= 0, \\ \theta(u - u(Q_1 + Q_2) + \mathcal{K}) &= 0, \\ \theta(u - u(R_1 + R_2) + \mathcal{K}) &= 0, \end{aligned} \tag{10}$$

in the Jacobian $Jac(\mathcal{X})$ is the union of $u(\mathcal{X})$– the image of the curve under AJ map – and eight points $u(P_j + Q_k + R_s),\ j, k, s = 1, 2$.

Proof. Again, we have to find the intersection of the solutions obtained in the previous theorem. Let us shift the same curve $u(\mathcal{X})$ by $u(P_s + Q_j)$ and by $u(P_m + R_k)$. The point u in this intersection has two representations

$$u = u(S + P_s + Q_j) = u(S' + P_m + R_k)$$

for some points S, S' of the curve. If $i(S + P_s + Q_j) = i(S' + P_m + R_k) = 1$, then two mentioned non-special divisors coincide: this may only happen in case $m = s$, $S = R_k$ and $S' = Q_j$ and therefore $u = u(P_s + Q_j + R_k)$. Assuming that speciality index is greater than 1, we come to a conclusion that $u \in u(\mathcal{X})$. ■

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