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Combinatorial description of a moduli space of curves and of extremal polynomials

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Abstract. For the description of extremal polynomials (that is, the typical solutions of least deviation problems) one uses real hyperelliptic curves. A partitioning of the moduli space of such curves into cells enumerated by trees is considered. As an application of these techniques the range of the period map of the universal cover of the moduli space is explicitly calculated. In addition, extremal polynomials are enumerated by weighted graphs.

Bibliography: 12 titles.

§ 1. History of the problem

We call a real polynomial $P(x)$ of one variable a (normalized) g-*extremal* polynomial if all but g of its critical points are simple, and it takes the values ± 1 at these points. The number of exceptional critical points can be calculated by the following formula:

$$g = \sum_{x:\ P(x)\neq\pm 1} \operatorname{ord} P'(x) + \sum_{x:\ P(x)=\pm 1} \left[\frac{1}{2}\operatorname{ord} P'(x)\right], \tag{1}$$

where $\operatorname{ord} P'(x)$ is the order of the zero of the derivative of P at $x \in \mathbb{C}$ and $[\,\cdot\,]$ is the integer part of a number. Extremal polynomials with $g = 0$ (Chebyshëv, 1853) or $g = 1$ (Zolotarëv, 1868) are well understood and used in the solution of many problems on the least deviation in the uniform metric [1]. General extremal polynomials allow one to extend considerably the spectrum of effectively solved optimization problems of this kind [2]. In the general case g-extremal polynomials are described in terms of real hyperelliptic curves of genus g and depend on g integer and g continuous parameters. Enumeration of the admissible systems of discrete parameters was the motivation behind the present paper.

For small g (depending on the computational powers available) extremal polynomials have been effectively calculated [3], [4] on the basis of their representation

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going back to Chebyshëv and thoroughly described in [2]. One characteristic feature of the Chebyshëv representation is the independence of the computational complexity of the degree n of the polynomial for fixed g. We now explain the idea of the representation, which is based on the geometric interpretation of Pell's equation with polynomial coefficient. This equation has been used for many years in the analysis of least deviation problems [5], and methods of its investigation go back to Abel's well-known paper of 1826. We associate with each polynomial $P(x)$ the hyperelliptic curve

$$M = M(\mathbf{e}) = \left\{(x,w) \in \mathbb{C}^2 : w^2 = \prod_{s=1}^{2g+2}(x-e_s)\right\} \tag{2}$$

with branch divisor $\mathbf{e} := \{e_s\}_{s=1}^{2g+2}$, where the e_s are the zeros of odd multiplicity of the polynomial $P^2(x)-1$. The genus g of this curve is equal to the number of exceptional critical points of $P(x)$ calculated by (1), with multiplicities taken into account. The polynomial $P(x)$ gives rise in natural fashion to a map[1] $\widetilde{P}(x,w)$ of the covering spaces in the diagram

$$\begin{array}{ccc} (x,w) \in M(\mathbf{e}) & \xrightarrow{\widetilde{P}} & \mathbb{CP}_1 \ni u \\ \downarrow \chi & & \downarrow \sigma \\ x \in \mathbb{CP}_1 & \xrightarrow{P} & \mathbb{CP}_1 \end{array} , \tag{3}$$

where $\chi(x,w) := x$ is the two-sheeted cover branched over the points in $\mathbf{e}$ and $\sigma(u) := \frac{1}{2}(u + \frac{1}{u})$ is the two-sheeted cover branched over ± 1. The map $\widetilde{P}(x,w)$ is equivariant with respect to the action of the covering transformation group (the transposition of the sheets): $\widetilde{P}(x,-w) = 1/\widetilde{P}(x,w)$, therefore the divisor of the function $\widetilde{P}(x,w)$ consists of two points, a pole of order $n := \deg P$ at the point at infinity ∞_+ on one sheet and a zero of the same order at the point at infinity ∞_- on the other sheet. The converse also holds: a meromorphic function $\widetilde{P}$ on $M(\mathbf{e})$ with divisor $n(\infty_- - \infty_+)$ satisfies (after normalization) the equivariance condition and lowers to a map $P(x)$ between the bases. This last map turns out to be a polynomial of degree n and necessarily has the required number of simple critical points with values ± 1 at these points.

In the framework of this construction the problem of the description of g-extremal polynomials of a fixed degree n is equivalent to enumeration of the curves M of the form (2) admitting a meromorphic function with divisor $n(\infty_- - \infty_+)$. The problem on the existence and a representation of such a function can be solved in terms of the curve M itself with the help of Abel's criterion [6]. There exists on each curve (2) a unique Abelian differential of the third kind

$$d\eta_M = \left(x^g + \sum_{s=0}^{g-1} c_s x^s\right)\frac{dx}{w} \tag{4}$$

[1] This map is also known as the Akhiezer function $P(x) + \sqrt{P^2(x)-1}$.

with purely imaginary periods. This normalization has a clear physical interpretation: one must put electric charges ± 1 at $\infty_\pm$; then the resulting potential on the Riemann surface M is the real part of the corresponding Abelian integral η_M. From the differential associated with M one can recover up to sign a polynomial $P(x)$ of degree n by the explicit formula

$$P(x) = \pm\cos\biggl(ni\int_{(e,0)}^{(x,w)} d\eta_M\biggr), \qquad x \in \mathbb{C}, \quad (x,w) \in M, \tag{5}$$

where the expression on the left-hand side is independent of the integration path, the ramification point $(e,0)$, or the point (x,w) on M lying over the independent variable of the polynomial. For the curve M associated with a polynomial of degree n the differential $d\eta_M$ is equal to $n^{-1}d\log\widetilde{P}(x,w)$, therefore the curves M determined by polynomials of degree n are precisely the ones with periods of the associated differential $d\eta_M$ belonging to the lattice $2\pi i n^{-1}\mathbb{Z}$.

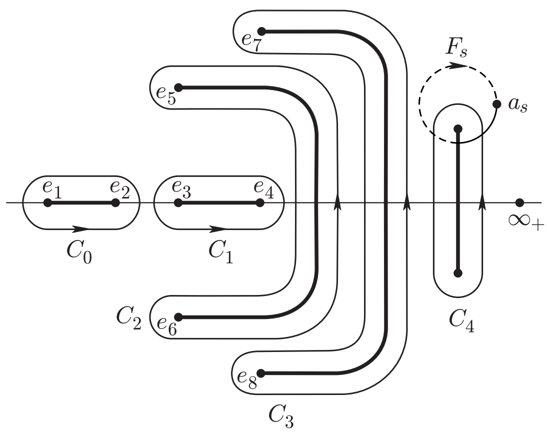


Figure 1. The system of cuts Λ in the plane for $g=4$, $k=2$ and an integral basis in $H_1^-(M,\mathbb{R})$

We consider in what follows only real polynomials $P(x)$. They are associated with *real curves* M, that is, ones admitting an *anticonformal involution* (reflection) $\overline{J}(x,w) := (\overline{x},\overline{w})$. The reflection $\overline{J}$ acts in the natural fashion in the *homology space* $H_1(M,\mathbb{R})$ of M, which can be obtained from the homology of the compact curve $M_c := M \cup \infty_+ \cup \infty_-$ by the addition of one generator, a cycle enclosing the puncture ∞_+ or ∞_-. A basis $C_0, C_1, \dots, C_g$ in the subspace of odd cycles $H_1^-(M,\mathbb{R}) := \{C \in H_1(M,\mathbb{R}) : \overline{J}C = -C\}$ is depicted in Fig. 1, where the thick lines depict cuts in the plane connecting pairwise the ramification points in M. The differential (4) associated with a real curve is itself real, that is, its coefficients c_s are real. The periods of the real differential $d\eta_M$ over even cycles surviving the reflection $\overline{J}$ are equal to zero and obviously belong to the lattice $2\pi i n^{-1}\mathbb{Z}$. The real curve M is associated with a real polynomial P of degree n if and only if the following system of Abel's equations holds [2]:

$$-i\int_{C_s} d\eta_M = 2\pi\frac{m_s}{n}, \qquad m_s \in \begin{cases} \mathbb{Z}, & s = 0,1,\dots,k-1, \\ 2\mathbb{Z}, & s = k,\dots,g; \end{cases} \tag{6}$$

here C_s is the fixed basis in the lattice of odd cycles and k is the number of coreal ovals in M. Some integers m_s must be even because after the projection onto even 1-cycles integral 1-cycles can become half-integral.

The left-hand sides of Abel's equations are locally single-valued analytic functions of the ramification points of the curve, but they are not globally well defined. If one transposes a pair of branch points in the upper half-plane, then one obtains another basis in the lattice of odd cycles. The left-hand sides of Abel's equations, the periods of the differential associated with the curve, live on the universal cover of the moduli space of real curves M, which is a $2g$-cell. Only g equalities in (6) are independent since the cycle $\sum_{s=0}^{g} C_s$ retracts to a pole with known residue of the differential $d\eta_M$. One obtains in this way the full-rank [2] *period map* $\Pi\colon \mathbb{R}^{2g} \to \mathbb{R}^g$. In the present paper we give an answer to two questions in connection with the period map:

(1) we find the image of the universal cover of the moduli space of real curves (2) under the period map;

(2) we present an example of a fibre of the period map that is invariant under the covering transformation group; this means that the projection of a fibre of the period map onto the moduli space is not an embedding in the general case.

§ 2. Main theorem

For the formulation of our main result we must give a rigorous definition of the period map of the universal cover of the moduli space of curves (2). Let us introduce the requisite concepts.

2.1. Moduli spaces of curves. Let $\mathbf{e}$ be an unordered system of distinct points $e_1, \dots, e_{2g+2}$ consisting of $2k$ real points and $g-k+1$ pairs of complex conjugate points. The group $\mathfrak{A}_1^+$ of orientation-preserving affine motions of the real axis acts freely on such systems: $\mathbf{e} = \{e_s\}_{s=1}^{2g+2} \to A\mathbf{e} + B = \{Ae_s + B\}_{s=1}^{2g+2}$, $A > 0, B \in \mathbb{R}$. We call the orbits of this action the *space* $\mathcal{H}_g^k$. Associating with each symmetric simple divisor $\mathbf{e}$ the hyperelliptic curve (2) we arrive at the equivalent definition of $\mathcal{H}_g^k$ as the *moduli space of (the conformal classes of) real hyperelliptic curves of genus g with k coreal ovals and a distinguished point ∞_+ on the oriented real oval that is distinct from the ramification points.*

The space $\mathcal{H}_g^k$ is a real $2g$-manifold. For the introduction of coordinates we number locally the points in the system $\mathbf{e}$ and fix a pair of complex conjugate or a pair of real points e_{2g+1}, e_{2g+2}. For local coordinates in the moduli space we take either the variables $\operatorname{Re} e_s$ in the case of real points e_s or $\operatorname{Re} e_s$, $\operatorname{Im} e_s$ for points e_s in the open upper half-plane $\mathbb{H}$, $1 \leqslant s \leqslant 2g$. Topologically, for $k > 0$ this space is the product of a $(2k-2)$-cell by the configuration space of the half-plane $\mathbb{H}$. In particular, the fundamental group $\pi_1(\mathcal{H}_g^k)$ is isomorphic [2] to the braid group on $g-k+1$ strands Br_{g-k+1} (this holds also for $k = 0$).

There exist several representations of the *universal cover* $\widetilde{\mathcal{H}}_g^k$ of the moduli space [4]; the most geometrically descriptive of them is the *labyrinth space*. A labyrinth $(\mathbf{e}, \Lambda)$ with invariants g and k is by definition a pair consisting of the above-described divisor $\mathbf{e}$ and a system $\Lambda = (\Lambda_0, \Lambda_1, \dots, \Lambda_g)$ of disjoint simple smooth arcs connecting pairwise points in the divisor (see Fig. 2). The first k arcs in the labyrinth are the projections of the coreal ovals $\{x \in \mathbb{R} : w^2(x) < 0\}$ of the curve $M(\mathbf{e})$. The other $g-k+1$ arcs connect complex conjugate points in $\mathbf{e}$, are invariant under the reflection relative to the real axis and bypass the first group of arcs on the right. The intersections with the real axis define a natural ordering of the arcs Λ_j. Two labyrinths $(\mathbf{e}, \Lambda)$ and $(\mathbf{e}', \Lambda')$ are said to be equivalent if there exists a motion in $\mathfrak{A}_1^+$ taking $\mathbf{e}$ to $\mathbf{e}'$ and the paths Λ to ones that can be continuously deformed into the paths Λ' so that the paths undergoing the deformation and the point set $\mathbf{e}'$ form a labyrinth at each instant. We call the set of labyrinths modulo this equivalence the labyrinth space $\mathcal{L}_g^k$.

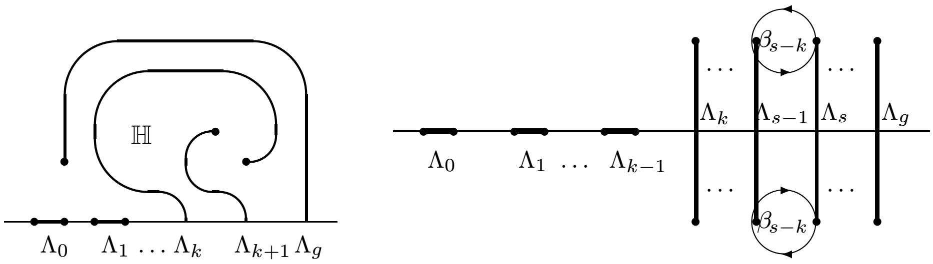


Figure 2. (a) $\mathbb{H}$ cut along Λ; (b) Dehn half-twists $\beta_1, \dots, \beta_{g-k}$ as covering transformations of $\mathcal{L}_g^k$

Theorem 1 [4]. *The labyrinth space $\mathcal{L}_g^k$ is homeomorphic to the universal cover $\widetilde{\mathcal{H}}_g^k$ of the moduli space and is a $2g$-cell.*

Both the labyrinth space and the universal cover of the moduli space are homeomorphic to the Teichmüller space of the disc with $g-k+1$ punctures and $2k+1$ marked points at the boundary. The Teichmüller space is just a $2g$-cell. A detailed proof of the above correspondences is outside the framework of the present paper as it requires special machinery. Informally, the universal cover of the moduli space is the set of branch divisors $\mathbf{e}$ supplied with the histories of their motion from the distinguished divisor $\mathbf{e}^0$. Regarding the cuts in a labyrinth as the traces of the motion of the divisor we can reconstruct this history.

The braid group Br_{g-k+1} acts by covering transformations in the universal cover of the moduli space and, in particular, in the labyrinth space $\mathcal{L}_g^k$. We realize Br_{g-k+1} as a modular group, the group $\mathrm{Mod}(\mathbb{H} \setminus \mathbf{e})$ of isotopy classes of diffeomorphisms of the punctured disc $\mathbb{H} \setminus \mathbf{e}$ fixed on its boundary [7]. Generators of this group are the Dehn half-twists $\beta_1, \beta_2, \dots, \beta_{g-k}$ along curves encircling a pair of punctures in the disc (see Fig. 2(b)). Extended by symmetry into the lower half-plane a transformation $f \in \mathrm{Mod}(\mathbb{H} \setminus \mathbf{e})$ takes the labyrinth $(\mathbf{e}, \Lambda)$ to the labyrinth $(\mathbf{e}, f\Lambda)$.

Our interest in labyrinths is explained by the fact that there exists no distinguished basis in the odd homology space on the curve $M(\mathbf{e})$, and a labyrinth $(\mathbf{e}, \Lambda)$ presents us with such a basis.

Definition. Going counterclockwise along the banks of the cuts $\Lambda_0, \Lambda_1, \dots, \Lambda_g$ on the upper sheet of $M(\mathbf{e})$ defines a *distinguished basis* $C_0, C_1, \dots, C_g$ in the space $H_1^-(M, \mathbb{R})$; see Fig. 1.

2.2. Transport of 1-cycles and period map. For the analysis of the periods of the differential $d\eta_M$ as functions of the variable curve M we construct the *homology bundle* [8] over the moduli space. The fibre over a point M of the rank $2g+1$ real vector bundle $H_1\mathcal{H}_g^k \to \mathcal{H}_g^k$ is the homology space $H_1(M,\mathbb{R})$ of M. Recall that M can be obtained from the compact curve by making two punctures at infinity, therefore one must add to the usual $2g$ independent 1-cycles the cycle encircling an (arbitrary) puncture. The homology bundle decomposes in the natural way into the sum of subbundles $H_1^+\mathcal{H}_g^k$ and $H_1^-\mathcal{H}_g^k$ with fibres that are the eigenspaces corresponding to the eigenvalues ±1 of the reflection operator $\overline{J}(x,w) := (\overline{x},\overline{w})$ on the homology. A local trivialization of the homology bundle is defined as follows.

Assume that a point $M(\mathbf{e})$ in the base space of the bundle lies in a sufficiently small neighbourhood of a fixed point $M_0 = M(\mathbf{e}^0)$ in the moduli space. Consider a plane diffeomorphism commuting with complex conjugation, taking the distinguished divisor $\mathbf{e}^0$ to a fixed divisor $\mathbf{e}$ and identical outside a small neighbourhood of points in the distinguished divisor. This map of the plane lifts to a map of its two-sheeted covers $M(\mathbf{e}^0) \to M(\mathbf{e})$ and induces an isomorphism of their homologies. This isomorphism $H_1(M(\mathbf{e}^0)) \simeq H_1(M(\mathbf{e}))$ respects the decomposition into even and odd 1-cycles and $\mathbb{Z}$-homology. We obtain a natural identification of the homologies at close points $\mathbf{e}$ and $\mathbf{e}^0$ in the moduli space, which can be used for a local trivialization of the homology bundle since it does not depend on one's choice of the plane diffeomorphism: all diffeomorphisms satisfying the above-mentioned conditions are isotopic in the punctured plane $\mathbb{C} \setminus \mathbf{e}^0$.

The sewing functions of our bundle are locally constant, therefore it possesses a natural flat (Gauss–Manin) connection [8]. This connection enables one to translate homology to nearby fibres, and its action on cycles in $H_1(M,\mathbb{Z})$ can be represented as follows. In the 2-sheeted model of the curve M we draw a closed contour representing a 1-cycle and avoiding the ramification points. Leaving this contour fixed and perturbing the ramification points we translate the 1-cycle to close curves M. This parallel translation of cycles is consistent with the above-mentioned decomposition of the homology bundle into the subbundles $H_1^{\pm}\mathcal{H}_g^k$ and preserves the lattice of $\mathbb{Z}$-homology. The distinguished basis corresponding to a labyrinth in § 2.1 is obviously respected by the parallel translation governed by the Gauss–Manin connection.

Example. The Dehn half-twist β_{s-k}, $s = k+1,\dots,g$, in Fig. 2(b) corresponds to an elementary braid in the group Br_{g-k+1} and to a loop in the moduli space. The translation of the distinguished basis of a labyrinth along this loop changes only two of its elements: $(C_{s-1},C_s)^t \to (-C_s, 2C_s+C_{s-1})^t$. In the process of deformation the cycle C_{s-1} moves to the lower sheet of the Riemann surface and dragging it to the upper sheet results in a change of sign. We see that the monodromy of the Gauss–Manin connection is non-trivial if so also is the topology of the moduli space.

The flat connection enables one to define an *action of the braid group* Br_{g-k+1} on the cohomology space $H^1(M_0,\mathbb{R})$ of the distinguished curve. Namely, we define the action of a loop $\beta \in \pi_1(\mathcal{H}_g^k, M_0)$ on a functional $C_* \in (H_1(M_0))^*$ by the following formula:

$$\langle \beta \cdot C_* \mid C\rangle := \langle C_* \mid \text{parallel translation of } C \text{ along } \beta\rangle, \qquad C \in H_1(M_0).$$

The *period map* Π takes the universal cover $\widetilde{\mathcal{H}}_g^k$ of the moduli space to functionals on the odd homology space of the distinguished curve by the following rule:

$$\langle \Pi(\widetilde{M}) \mid C\rangle := -i\int_C d\eta_M, \qquad \widetilde{M}\in\widetilde{\mathcal{H}}_g^k, \quad C\in H_1^-(M_0,\mathbb{R}). \tag{7}$$

Here integration of the differential on M over cycles in the distinguished (and therefore distinct from M in the general case) curve M_0 proceeds with the use of the Gauss–Manin connection in the homology bundle over the universal cover $\widetilde{\mathcal{H}}_g^k$. The following result holds.

Lemma 1 [2]. *The period map* $\Pi\colon \widetilde{\mathcal{H}}_g^k\to (H_1^-(M_0,\mathbb{R}))^*$ *commutes with the action of the braid group* Br_{g-k+1}.

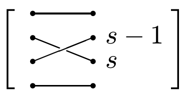
We describe explicitly the range of the period map in Theorem 2, for the statement of which we require several definitions.

2.3. Statement of the main theorem. We distinguish a point $\widetilde{M}_0=(\mathbf{e}^0,\Lambda^0)$ in the universal cover $\widetilde{\mathcal{H}}_g^k$. The distinguished basis $C_0,C_1,\dots,C_g$ of the labyrinth Λ^0 introduced in § 2.1 provides us with the system of coordinates $\gamma_s:=\langle\cdot\mid C_s\rangle$ in the space of functionals over the odd 1-cycles $(H_1^-(M_0,\mathbb{R}))^*\cong\mathbb{R}^{g+1}$. The hyperplane $\{\gamma_0+\gamma_1+\dots+\gamma_g=2\pi\}$ in that space contains the image of $\widetilde{\mathcal{H}}_g^k$ under the period map Π.

We fix topological invariants g and k, $g\geqslant 0$, $0<k\leqslant g+1$. The points $h:=(h_0,h_1,\dots,h_{g+1})^t$ in the Euclidean space $\mathbb{R}^{g+2}$ with coordinates satisfying the conditions

$$0=h_0<h_1<h_2<\dots<h_g<h_{g+1}=\pi, \tag{8}$$

fill an open g-simplex Δ_g. We define as follows the action of the braid group Br_{g+2} in the ambient space on the generators (Burau, 1932):



$$\cdot(\dots,h_{s-1},h_s,\dots)^t:=(\dots,h_s,2h_s-h_{s-1},\dots)^t, \tag{9}$$

$s=1,2,\dots,g+1$; a crossing on the $(s-1)$th and sth strands affects only the variables h_{s-1} and h_s.

With each $(g-k+1)$-element subset $\mathfrak{i}$ of the index set $\{1,2,\dots,g\}$ we associate the braid $\varkappa(\mathfrak{i})\in\mathrm{Br}_{g+2}$ (see Fig. 3). We move the strands of the trivial braid with indices in $\mathfrak{i}$ away from the plane containing the figure, and fixing their right-hand end-points shift them upwards without crossings, while the other strands we shift downwards without crossings. The braid $\varkappa(\mathfrak{i})$ acts as follows at a point $h\in\mathbb{R}^{g+2}$: the variables h_s, $s\in\mathfrak{i}$, take the first $g-k+1$ positions in the order of their indices s. The other variables h_j, $j\notin\mathfrak{i}$, undergo successive reflections at all the points h_s, $s\in\mathfrak{i}$, with larger indices $s>j$ and after these transformations assume the last $k+1$ positions in the order of their indices j.

We treat braids β on $g-k+1$ strands as embedded in Br_{g+2} by adding to them horizontal strands below; see Fig. 3.

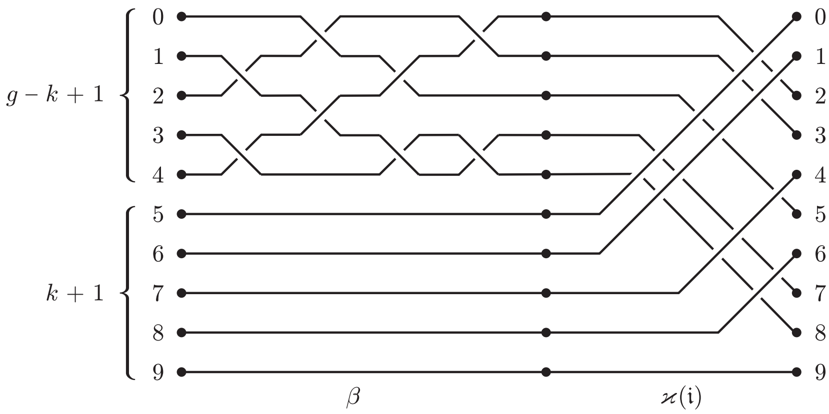


Figure 3. The braids $\beta \in \mathrm{Br}_{g-k+1}$ and $\varkappa(\mathfrak{i}) \in \mathrm{Br}_{g+2}$ for $g = 8$, $k = 4$, and $\mathfrak{i} = \{2, 3, 5, 7, 8\}$

Theorem 2. *The set* $\Pi(\widetilde{\mathcal{H}}_g^k)$, $k > 0$, *is the image under the following linear map* $\mathbb{R}^{g+2} \to \mathbb{R}^{g+1}$ *of the union in* $\mathbb{R}^{g+2}$ *of the open simplexes* $\beta \cdot \varkappa(\mathfrak{i}) \cdot \Delta_g$ *taken over all* $\binom{k-1}{g}$*-subsets of* $\mathfrak{i}$ *and all braids* β *in* Br_{g-k+1}:

$$\gamma_s := \begin{cases} 2(h_{g-s+1} - h_{g-s}), & s = 0, \dots, k-1, \\ 4(-1)^{g+s} h_{s-k}, & s = k, \dots, g. \end{cases} \tag{10}$$

Example. We depict the 2- and 3-dimensional images $\Pi(\widetilde{\mathcal{H}}_g^k)$ in Figs. 4 and 5.

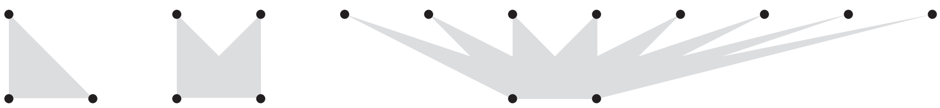


Figure 4. The range of the period map $\Pi(\widetilde{\mathcal{H}}_2^k)$ for $k = 3, 2, 1$ (from left to right)

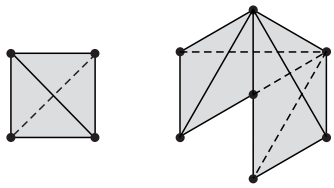


Figure 5. The range of the period map $\Pi(\widetilde{\mathcal{H}}_3^k)$ for $k = 4, 3$ (from left to right)

§ 3. Partitioning the moduli space into cells

We develop below a combinatorial geometric approach to the study of the period map. To curves M from the moduli space we shall assign in a one-to-one fashion trees Γ of a special form with edges labelled by positive numbers. This defines a partitioning of the total moduli space $\mathcal{H} := \coprod_{g=0}^{\infty} \coprod_{k=0}^{g+1} \mathcal{H}_g^k$ into cells $\mathcal{A}[\Gamma]$ enumerated by trees, in which some of the global coordinate variables are periods of the differential associated with the curve. Similar cell partitionings of the moduli spaces of curves are used in conformal field theory (Penner, Kontsevich, Chekhov; see the references in the survey paper [9]). We present applications of these techniques in the last two sections of this paper.

3.1. Curves and trees. The Abelian integral of the third kind $\eta_M(x) := \int^x d\eta_M$ is a Schwarz–Christoffel integral mapping the suitably cut upper half-plane $\mathbb{H}$ onto a certain comb-like domain with geometric parameters allowing one to recover all the periods of $d\eta_M$. This important observation was made for the first time for a special case by Akhiezer (see a discussion in [3]) and in a more general situation by Meĭman [10]. To describe the system of cuts one can conveniently use Strebel's theory of foliations associated with quadratic differentials [11]. Throughout § 3.1 the curve M is fixed, therefore we drop the corresponding subscript of the associated differential $d\eta_M$.

3.1.1. Global width function and foliation. In view of the normalization of $d\eta$, the function

$$W(x) := \left| \operatorname{Re} \int_{(e_s,0)}^{(x,w)} d\eta \right|$$

is well defined in the entire x-plane, vanishes at all the branch points $x = e_j$, $j = 1, \dots, 2g+2$, has a logarithmic pole at infinity and is harmonic outside its zero set. In other words, the *width function* $W(x)$ is the Green's function of the plane cut along the piecewise smooth set $\Gamma_{\mathsf{I}} := \{x \in \mathbb{C} : W(x) = 0\}$ containing all branch points. The foliation $(d\eta)^2(x) > 0$ is orthogonal to the level curves of $W(x)$ and therefore has neither cycles nor limit cycles. In Strebel's terminology [11] the global structure of the trajectories in the foliation is as follows: the leaves starting at finite *critical points* (that is, at points in the divisor of $(d\eta)^2$) partition the complex plane into strips each coming from infinity, passing through a component of the zero level of $W(x)$, and returning to infinity. The local structure of the foliation $(d\eta)^2(x) > 0$ in the neighbourhood of its critical points is depicted in Fig. 6.

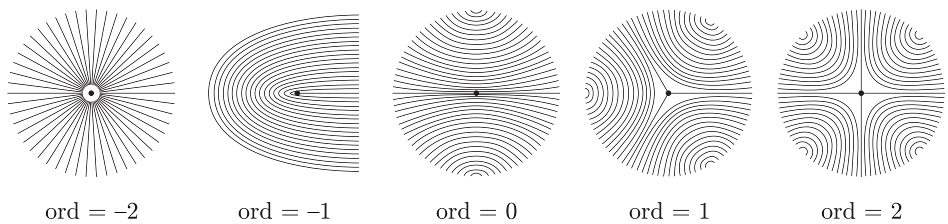


Figure 6. The foliation $(d\eta)^2 > 0$ in the neighbourhood of points with $\operatorname{ord}(d\eta)^2 = -2, -1, 0, 1, 2$

3.1.2. The graph Γ of the curve M.

Construction. We associate with each curve $M \in \mathcal{H}$ a compact planar graph $\Gamma = \Gamma(M)$ consisting of two parts, $\Gamma_{|}$ and $\Gamma_{\_}$. The component $\Gamma_{|} := \{x \in \mathbb{C} : W(x) = 0\}$ is said to be *vertical.* By the *horizontal* component $\Gamma_{\_}$ we shall mean the set of intervals of the foliation $(d\eta)^2(x) > 0$ connecting critical points of the foliation with one another or with $\Gamma_{|}$ and oriented in the increasing direction of $W(x)$. By *vertices* V of the graph Γ we mean all finite points of the divisor of $(d\eta)^2$ and points in the intersection $\Gamma_{|} \cap \Gamma_{\_}$. Outside its vertices the graph is the union of finitely many intervals of vertical and horizontal trajectories of the quadratic differential $(d\eta)^2$, which we call its edges R. The measures $|\mathrm{Re}(d\eta)|$ and $|\mathrm{Im}(d\eta)|$ connected with $(d\eta)^2$ allow one to define the *width* W and the *height* H, respectively, of an arbitrary smooth curve in the plane. For instance, the width $W(R)$ of an edge $R \subset \Gamma$ is equal to the increase of the width function $W(x)$ along R, and the height $H(R)$ of each horizontal edge R is zero.

Remark. In terms of generic polynomials $P(x)$ the graph Γ was introduced for the first time in [10]. Its part $\Gamma_{|} := P^{-1}([-1, 1])$ had been considered before by many authors (see the references in [3]) in their studies of polynomials of least deviation on several intervals and is called the *support set of the polynomial*, the *n-regular set*, *or* the *maximal least deviation set.*

Thus, each real hyperelliptic curve gives rise to a planar graph Γ with edges of two types: oriented horizontal edges equipped with their widths W, and non-oriented vertical edges equipped with their heights H. We present several typical graphs $\Gamma(M)$ in Figs. 7(a), 9(b), 10, 11(a), where *the double line denotes the vertical component.* In what follows we are interested in a graph only as a combinatorial object equipped with numbers. We say that two partially oriented graphs in the plane are equivalent if one of them can be transformed into the other by an orientation-preserving motion of $\mathbb{H}$ extended by symmetry into the entire plane. We denote the equivalence class of a graph Γ by $[\Gamma]$, and the equivalence class of a graph with edges labelled by positive numbers by $\{\Gamma\}$.

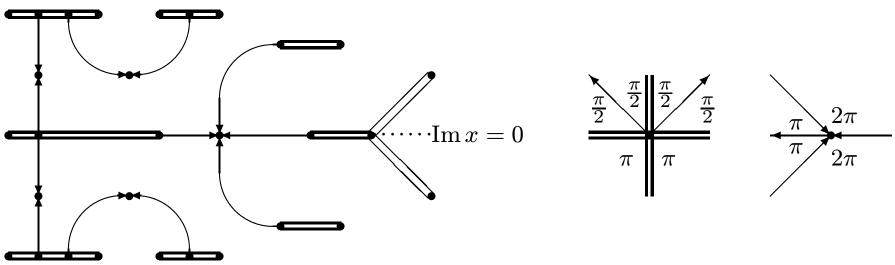


Figure 7. (a) Admissible graph of a curve $M \in \mathcal{H}_8^2$; (b) calculation of the order of $(d\eta)^2$ at a vertex of a graph

3.1.3. Functions of a graph $\Gamma(M)$.

Let us introduce several combinatorial characteristics of a graph whose meaning will be clear from what follows. We denote the subgraphs of Γ lying in the upper half-plane or on the real axis by $\Gamma_\star^+ := \Gamma_\star \cap \mathbb{H}$ and $\Gamma_\star^0 := \Gamma_\star \cap \mathbb{R}$, respectively; the index $\star$ can be empty or equal to $|$ or $\_$. The degree of a vertex V relative to the graph $\Gamma^\bullet_\star$ will be denoted by $d^\bullet_\star = d^\bullet_\star(V)$. The horizontal edges are oriented, therefore one can define also the quantities $d^\bullet_{\mathrm{in}}(V)$ and $d^\bullet_{\mathrm{out}}(V)$ equal to the numbers of edges of the graph $\Gamma^\bullet_{-}$ starting or ending at the vertex V, respectively, where the index $\bullet$ can be empty or equal to 0 or $+$.

We now introduce further notation:

$$\begin{aligned}
\mathrm{ord}(V) &:= d_{|}(V) + 2d_{\mathrm{in}}(V) - 2,\\
g(\Gamma) &:= \#\{V \in \Gamma : \mathrm{ord}(V) \equiv 1 \,(\mathrm{mod}\, 2)\}/2 - 1,\\
k(\Gamma) &:= \#\{V \in \Gamma^0 : \mathrm{ord}(V) \equiv 1 \,(\mathrm{mod}\, 2)\}/2,\\
\dim(\Gamma) &:= \#\{R \subset (\Gamma^+_{|} \cup \Gamma^0_{|})\} + \#\{V \in (\Gamma^+_{-} \cup \Gamma^0_{-}) \setminus \Gamma_{|}\} - 1,\\
\mathrm{codim}(\Gamma) &:= 2g(\Gamma) - \dim(\Gamma),\\
\mathrm{codim}(V) &:= d^+_{\mathrm{out}}(V) + \begin{cases} 2d_{\mathrm{in}}(V) - 4, & V \in \Gamma^+_{-} \setminus \Gamma^+_{|},\\ d_{\mathrm{in}}(V) - 2, & V \in \Gamma^0_{-} \setminus \Gamma^0_{|},\\ 2(d_{|}(V)\,(\mathrm{mod}\, 2)) + d^+_{|}(V) - 3, & V \in \Gamma^+_{|},\\ d_{|}(V)\,(\mathrm{mod}\, 2) + d^+_{|}(V) - 1, & V \in \Gamma^0_{|}; \end{cases}
\end{aligned}$$

in the last definition $V \in \Gamma^0 \cup \Gamma^+$ and the residue mod 2 is 0 or 1.

3.1.4. Properties of the graph $\Gamma(M)$. In terms of the original curve M the combinatorial characteristics ord, g, and k are explained by the following result.

Lemma 2. *Let $\Gamma(M)$ be a graph corresponding to a curve $M \in \mathcal{H}^k_g$. Then*

(1) $\mathrm{ord}(V)$ *is the order of the quadratic differential $(d\eta)^2$ at the vertex V*,
(2) $g(\Gamma)$ *is the genus of M*,
(3) $k(\Gamma)$ *is the number of coreal ovals on M.*

Proof. (1) The quantity $2 + \mathrm{ord}(V)$, which is either $d_{|}$ for $W(V) = 0$ or $2d_{\mathrm{in}}$ for $W(V) > 0$, has the meaning of the number of trajectories in the (vertical or horizontal) foliation of the quadratic differential $(d\eta)^2$ that are incident to V, therefore [11] this integer is larger by 2 than the order of $(d\eta)^2$ at this vertex.

(2) Points at which the order of $(d\eta)^2$ is odd are ramification points in M.

(3) $2k(M)$ is the number of ramification points of M lying over the real axis.

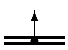
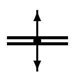
For weighted graphs $\{\Gamma\}$ associated with curves there exist several structural and qualitative constraints, which are listed in the next statement.

Lemma 3. (A$_1$) $[\Gamma]$ *is a finite tree symmetric* (*together with all its weight structures*) *relative to the real axis.*

(A$_2$) *Horizontal edges starting at some vertex are separated by edges ending at it or by vertical edges. In particular, forbidden are hanging vertices of the following form*:

 .

(A$_3$) *Vertices with* $\mathrm{ord}(V) = 0$ *can be only of two kinds*:

 *and*  .

(A$_4$) *The sum of the heights H of all vertical edges in a tree is π.*
(A$_5$) *If $V \in \Gamma_{|}$, then $W(V) = 0$.*

Proof. (A_1) We claim that Γ is a connected graph without cycles. Assume that it contains cycles; then there exists a bounded region in its complement. Integrating the square of the gradient of $W(x)$ over it we obtain 0 because W is a harmonic function outside Γ vanishing at the vertical edges of the graph, while its normal derivative vanishes at the horizontal edges.

The graph is connected since the sum $\operatorname{ord}(V)$ taken over all vertices is twice the number of edges minus twice the number of vertices, that is, it is equal to the number of components of Γ multiplied by -2. By Lemma 2 this is also equal to the degree of the divisor of the quadratic differential $(d\eta)^2$ on the sphere plus the order of its pole at infinity, that is, -2. The symmetry of the weighted graph can be seen from the fact that the quadratic differential is real: $(d\eta)^2(\overline{x}) = (d\overline{\eta})^2(x)$.

(A_2) If $W(V) > 0$, then the function W alternately increases and decreases on the horizontal trajectories of $(d\eta)^2$ with end-points at V. Trajectories of the second type are locally the same as horizontal edges of Γ ending at V. On the other hand, if $W(V) = 0$, then the horizontal trajectories starting at V alternate with vertical trajectories, which coincide locally with vertical edges of Γ.

(A_3) $\operatorname{ord}(V) = 0$ only if $V \in \Gamma_{|} \cap \Gamma_{-}$.

(A_4) In the domain $\mathbb{C} \setminus \Gamma_{|}$ the differential $d\eta(x)$ has a single-valued branch since each component of $\Gamma_{|}$ contains an *even* number of branch points of M. Deforming the boundary of the domain into a large circle we obtain

$$\pm 2\pi i = \int_{\partial(\mathbb{C}\setminus\Gamma_{|})} d\eta = i \int_{\partial(\mathbb{C}\setminus\Gamma_{|})} \operatorname{Im}(d\eta) = \pm 2i \int_{\Gamma_{|}} |\operatorname{Im}(d\eta)| = \pm 2i \sum_{R\subset\Gamma_{|}} H(R).$$

(A_5) This holds by definition.

The next lemma gives one a hint of the method of the recovery of a hyperelliptic curve from a weighted graph.

Lemma 4 [10]. *The Abelian integral of the 3rd kind $\eta(x)$ maps homeomorphically the upper half-plane cut along Γ onto a half-strip of height π with finitely many cuts starting at the upper part of its boundary.*

Proof. The Abelian integral has the following form: $\eta(x) = W(x) + iH(x)$, $x \in \mathbb{H}\setminus\Gamma$, where $W(x)$ is the global width function and $H(x)$ is the conjugate harmonic function with a single-valued branch in the simply-connected domain $\mathbb{H} \setminus \Gamma$ normalized by the condition $H(x) = 0$, $x > \Gamma^0$. We shall interpret the height function $H(x)$ in terms of the foliation $(d\eta)^2 > 0$ restricted to the domain $\mathbb{H} \setminus \Gamma$.

Each leaf of the foliation starts and ends at the boundary of the domain, which consists of the following parts: ∞, $\mathbb{R} \setminus \Gamma$, Γ_{-}, $\Gamma_{|}$. Searching through all possible cases we can derive the following behaviour of the leaves oriented in the increasing direction of $W(x)$: *finitely many leaves start at vertices of Γ_{-}, all other leaves start from $\Gamma_{|}$; all leaves go to infinity staying within the upper half-plane.* For instance, a leaf cannot start or end at $\mathbb{R} \setminus \Gamma$; otherwise its end-point is a critical point of the foliation (since the foliation is mirror-symmetric) and therefore lies in Γ.

The function H is constant on each leaf and is equal to $\lim \operatorname{Arg}(x)$ as $x \to \infty$ along the leaf. Corresponding to each value of $H \in (0, \pi)$ there exists a unique leaf in the cut half-plane and the width function W allows one to distinguish points in this leaf. Hence the function $\eta(x)$ maps $\mathbb{H} \setminus \Gamma$ one-to-one onto a comb-like domain.

3.1.5. Recovery of the curve M from its graph Γ. Let $\{\Gamma\}$ be a tree in the plane with edges of two types, non-oriented 'vertical' and oriented 'horizontal' ones. We call the positive numbers labelling the edges R their widths $W(R)$ if $R \subset \Gamma_{\_}$ or their heights $H(R)$ if $R \subset \Gamma_{|}$. We set the height of horizontal edges and the width of vertical ones to be zero. On the vertices V of the tree we can (uniquely in effect) introduce the width function $W(V)$ strictly increasing on (oriented) horizontal edges such that the width of each edge is equal to the increment of the width function between its end-points. If the tree satisfies conditions (A_1)–(A_5), then we can associate with it a real hyperelliptic curve $M = M\{\Gamma\}$.

Construction. It follows from (A_1) that $\mathbb{H} \setminus \Gamma$ is a simply-connected domain. Each piece of its boundary inherits from $\{\Gamma\}$ the notions of vertices v, vertical and horizontal edges r, height function $H(r)$, and width function $W(v)$. Going along the boundary of the cut half-plane from $-\infty$ to $+\infty$ defines a strict ordering of the vertices v and the edges r, so that on the vertices in the boundary one can define the height function

$$H(v) := \sum_{r>v} H(r).$$

Conditions (A_2) and (A_5) ensure that the monotonic behaviour of the height and the weight functions of vertices v under the motion in the positive direction along the boundary $\mathbb{H} \setminus \Gamma$ is similar to Fig. 8.

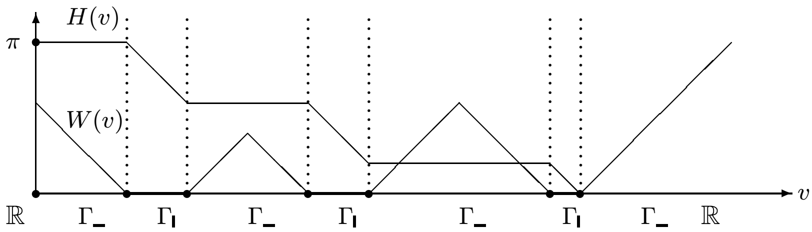


Figure 8. Monotonicity of $H(v)$ and $W(v)$ on the boundary of $\mathbb{H} \setminus \Gamma$

In the η-plane we consider now the comb-like domain

$$\Sigma\{\Gamma\} := \{\eta \in \mathbb{C} : 0 < \operatorname{Re}\eta,\ 0 < \operatorname{Im}\eta < \pi\} \setminus \bigcup_{v \in \partial(\mathbb{H}\setminus\Gamma)} \Bigl[iH(v), iH(v) + \max_{H(v')=H(v)} W(v')\Bigr], \tag{11}$$

and map the vertices in the boundary of $\mathbb{H} \setminus \Gamma$ into its boundary by the formula

$$v \to \eta(v) := W(v) + iH(v) \pm i0, \tag{12}$$

where $\pm$ is the sign of the monotonicity of W at the vertex v. It is clear from the graph in Fig. 8 that the map between the boundaries is monotonic relative to the ordering of the vertices, therefore it associates in a unique fashion edges r in the boundary of $\mathbb{H} \setminus \Gamma$ with edges $\eta(r)$ in $\partial\Sigma\{\Gamma\}$. Corresponding to each edge $R \subset \Gamma^+$

in the boundary of the half-plane cut along Γ there is a pair of edges r^-, r^+: its banks. The linear map of the interval $\eta(r^-)$ onto $\eta(r^+)$ reversing the orientation of the boundary of the comb has the following form:

$$\eta \to \begin{cases} \eta + iH^+ - iH^-, & R \subset \Gamma_{-}^{+}, \\ -\eta + iH^+ + iH^-, & R \subset \Gamma_{|}^{+}, \end{cases} \tag{13}$$

where $H^\pm$ is the mean value of the height $H = \operatorname{Im}\eta$ on the interval $\eta(r^\pm)$. We shall treat the intervals $\eta(r^-)$ and $\eta(r^+)$ as identified in accordance with (13).

Taken with all boundary identifications the comb-like domain $\Sigma\{\Gamma\}$ is an abstract Riemann surface $\mathbb{H}\{\Gamma\}$ homeomorphic to a disc. A conformal map $x(\eta)\colon \mathbb{H}\{\Gamma\} \to \mathbb{H}$ with fixed infinity is defined up to motions in $\mathfrak{A}_1^+$. It is clear from gluing formula (13) that on the half-plane $\mathbb{H}$ we obtain a well-defined meromorphic quadratic differential $(d\eta(x))^2$. It is real at the boundary and extends by symmetry into the lower half-plane. We take for $M\{\Gamma\}$ the two-sheeted surface (2) with branch points at the zeros (the poles) of odd multiplicity of the quadratic differential $(d\eta)^2(x)$, $x \in \mathbb{C}$.

Theorem 3. *The constructions in* §§ 3.1.2 *and* 3.1.5 *establish a one-to-one correspondence between hyperelliptic curves* $M \in \mathcal{H}$ *and weighted graphs* $\{\Gamma\}$ *satisfying conditions* (A_1)–(A_5).

Proof. Let $\{\Gamma\} \to M\{\Gamma\} \to \Gamma'(M)$ be the maps from §§ 3.1.5 and 3.1.2 performed successively. The construction of the first map associates with an abstract graph $\{\Gamma\}$ its realization $x(\Gamma)$ in the x-plane with vertices $x(V)$, smooth edges $x(R)$, and distinct subgraphs $x(\Gamma_{|})$, $x(\Gamma_{-})$, $\dots$. We claim that the weighted graphs $x(\Gamma)$ and Γ' are equal.

A calculation of angles at all the vertices $\eta(v)$ in the boundary of the comb $\Sigma\{\Gamma\}$ corresponding to a vertex $V \in [\Gamma]$ shows (see Fig. 7(b)) that $\operatorname{ord}(V)$ is equal to the order of $(d\eta)^2$ at $x(V)$. This means that all branch points of the curve $M\{\Gamma\}$ are end-points of edges in $x(\Gamma_{|})$. Each connected component of $x(\Gamma_{|})$ contains finitely many branch points, therefore $M\{\Gamma\}$ is realized as a pair of sheets $\mathbb{C} \setminus x(\Gamma_{|})$ glued crosswise along the edges in $x(\Gamma_{|})$. The identifications (13) show that $d\eta(x)$ on the upper sheet and $-d\eta(x)$ on the lower one are glued into a single meromorphic differential on M with two simple poles at infinity. We shall show that this is the differential $d\eta_M$ associated with the curve M by calculating its periods.

Assume that $\mathbb{R} \cup x(\Gamma)$ intersects transversally the projection of a cycle $C \subset M$ onto the x-plane partitioning this projection into pieces $C_1, C_2, C_3, \dots$. We reverse the orientation of the pieces of C corresponding to the lower sheet of M; then the integral of $d\eta$ over C is equal to the sum of the increments of the argument of η along the images of the pieces $C_1, C_2, C_3, \dots$ in the two combs $\Sigma\{\Gamma\} \cup \overline{\Sigma\{\Gamma\}}$. Each intersection of the projection of the cycle C with $\mathbb{R} \cup x(\Gamma_{-})$ yields a pair of points on the horizontal part of the boundary of the combs, which enter the above-mentioned alternating sum with opposite signs. Each intersection with $x(\Gamma_{|})$ yields a pair of points in the vertical part of the boundary, which enter this sum with the same sign. The rules (13) of the identification of the boundary edges of the combs now allow us to say that the integral of $d\eta$ over an arbitrary closed cycle C in M is an integer linear combination of the quantities $iH(R)$, $R \subset \Gamma_{|}$. In particular, all the periods of the differential are purely imaginary. Condition (A_4) enables one to calculate the polar period: the residue of $d\eta$ at the point at infinity in the upper sheet is -1.

The remaining verification of the equality of the weighted graphs $\Gamma'(M)$ and $x(\Gamma)$ is routine. Condition (A_3), which we have not used yet, guarantees that $x(\Gamma)$ has no vertices distinct from finite points in the divisor of $(d\eta)^2$ or in $x(\Gamma_{|}) \cap x(\Gamma_{-})$.

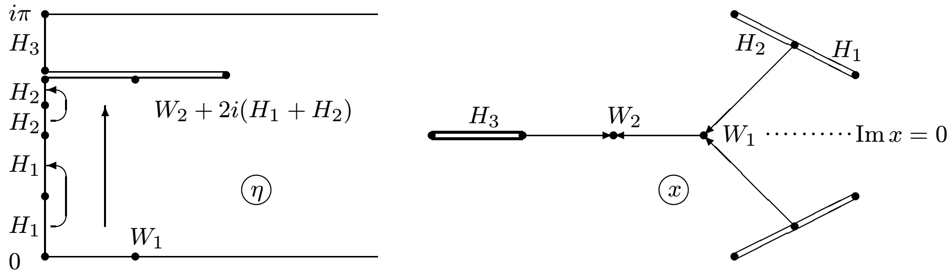


Figure 9. (a) A comb-like domain with identifications of edges; (b) a weighted tree Γ

Conversely, let $M \to \Gamma(M) \to M'\{\Gamma\}$ be the maps of §§ 3.1.2 and 3.1.5 performed successively. The conformal map $x(\eta)$ in the second construction was the inverse of the Abelian integral $\eta(x)$ in Lemma 4 — it is in this way that one obtains the map (12) and the gluings (13). Hence $M = M'$ up to an affine motion from $\mathfrak{A}_1^+$.

3.2. Coordinate space of a graph. We fix a topological class $[\Gamma]$ of graphs. The systems of weights on a graph satisfying conditions (A_1)–(A_5) satisfy the following constraints:

$$H(R) > 0, \qquad \sum_{R \subset \Gamma_{|}^0} H(R) + 2 \sum_{R \subset \Gamma_{|}^+} H(R) = \pi, \ \text{ the edge } \ R \subset \Gamma_{|}^+ \cup \Gamma_{|}^0; \tag{14}$$

$$W(V) > 0, \quad W(V') > W(V) \ \text{ for } \ V' > V, V, V' \ \text{ are vertices in } (\Gamma_{-}^+ \cup \Gamma_{-}^0) \setminus \Gamma_{|}; \tag{15}$$

recall that the vertices in the horizontal part Γ_{-} of the graph are partially ordered. This set of numbers parametrizes some manifold, depending on $[\Gamma]$, in the total moduli space $\mathcal{H}$, which therefore is partitioned into disjoint manifolds enumerated by admissible topological types $[\Gamma]$.

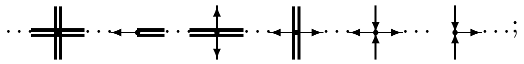
Definition. By the *coordinate space* $\mathcal{A}[\Gamma]$ of a graph Γ we shall mean the product of the open simplex (14) filled by the values of the variables $H(R)$ and the open cone (15) filled by the values of the variables $W(V)$.

Example. For the graph in Fig. 9(b) the coordinate space is an open 4-dimensional polyhedron in the space $\mathbb{R}^5 \ni (H_1, H_2, H_3, W_1, W_2)$ described by the constraints $2(H_1 + H_2) + H_3 = \pi$; $H_s > 0$, $s = 1, 2, 3$; $W_2 > W_1 > 0$.

We have the following result.

Lemma 5. *The dimension* $\dim(\Gamma)$ *of a cell* $\mathcal{A}[\Gamma]$ *does not exceed* $2g(\Gamma)$ *and is equal to* $2g$ *if and only if all vertices* $V \in \Gamma$ *have the following form*:

and the vertices $V \in \Gamma^0$ *on the real axis can, in addition, have the following form* (*up to the symmetry* $x \to -x$):



the axis $\mathbb{R}$ *is plotted by dots.*

Proof. We can show by brute force that the defect of a vertex $\operatorname{codim}(V)$ is nonnegative for $V \in \Gamma^0 \cup \Gamma^+$, and only for vertices of the above-described form $\operatorname{codim}(V) = 0$. The calculation below demonstrates that the sum of the defects of the vertices V is equal to the codimension of the cell $\mathcal{A}[\Gamma]$ in the corresponding moduli space $\mathcal{H}_g^k$:

$$
\begin{aligned}
\sum_{V\in\Gamma^0\cup\Gamma^+} \operatorname{codim}(V) = &\sum_{V\in\Gamma^0\cup\Gamma^+} d^+_{\text{out}} + 2\sum_{V\in\Gamma^+_{-}\setminus\Gamma_{|}} (d_{\text{in}}-2) + \sum_{V\in\Gamma^0_{-}\setminus\Gamma_{|}} (d_{\text{in}}-2) \\
&+ \sum_{V\in\Gamma^+_{|}} (2(d_{|}\,(\operatorname{mod} 2)) + d^+_{|} - 3) + \sum_{V\in\Gamma^0_{|}} (d_{|}\,(\operatorname{mod} 2) + d^+_{|} - 1) \\
= &(2g(\Gamma)+2) + \sum_{R\subset\Gamma^+_{-}} (1+2) + \sum_{R\subset\Gamma^0_{-}} 1 + \sum_{R\subset\Gamma^+_{|}} 2 \\
&- \sum_{V\in\Gamma^+_{-}\setminus\Gamma_{|}} 4 - \sum_{V\in\Gamma^0_{-}\setminus\Gamma_{|}} 2 - \sum_{V\in\Gamma^+_{|}} 3 - \sum_{V\in\Gamma^0_{|}} 1 \\
= &(2g(\Gamma)+2) + \Big\{\sum_{R\subset\Gamma^+} 3 - \sum_{V\in\Gamma^+} 3\Big\} + \Big\{\sum_{R\subset\Gamma^0} 1 - \sum_{V\in\Gamma^0} 1\Big\} \\
&- \sum_{R\subset\Gamma^+_{|}} 1 - \sum_{R\subset\Gamma^0_{|}} 1 - \sum_{V\in\Gamma^+_{-}\setminus\Gamma_{|}} 1 - \sum_{V\in\Gamma^0_{-}\setminus\Gamma_{|}} 1 \\
= &\,2g(\Gamma) - \dim(\Gamma).
\end{aligned}
$$

In the last equality we take into account the fact that the expression in the first curly brackets is equal to zero (each component of Γ^+ is a tree with one vertex removed) and the expression in the second curly brackets is equal to -1 (Γ^0 is a tree).

Example. Corresponding to the moduli space $\mathcal{H}_3^2$ there are 20 graphs $[\Gamma]$ with maximum dimension of the coordinate space $\dim(\Gamma) = 6$. We present them in Fig. 10 up to the symmetry relative to the vertical axis.

Theorem 4. *The coordinate space* $\mathcal{A}[\Gamma]$ *is real-analytically embedded in the total moduli space* $\mathcal{H}$.

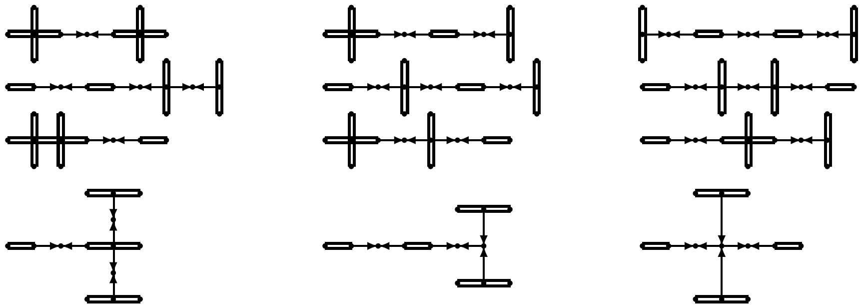


Figure 10. Stable graphs corresponding to the space $\mathcal{H}_3^2$

Proof. Each component of the moduli space $\mathcal{H}$ is stratified into smooth strata in accordance with the multiplicities of the zeros of the associated differential:

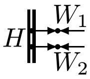
$$d\eta_M = \prod_{s=1}^{2g+2} (x-e_s)^{\varepsilon_s} \prod_{j=1}^{l} (x-a_j)^{\alpha_j} \frac{dx}{w}, \qquad \varepsilon_s \geqslant 0, \quad \alpha_j \geqslant 1, \quad \sum_{s=1}^{2g+2} \varepsilon_s + \sum_{j=1}^{l} \alpha_j = g, \tag{16}$$

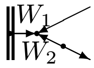
where all the zeros $a_j \neq e_s$ of the differential are distinct and form a symmetric set relative to the real axis. On such a stratum of real dimension $g + l$ one can introduce local coordinates $\gamma_1, \dots, \gamma_g$; $\varphi_1, \dots, \varphi_l$ as follows [2]:

$$\gamma_s(M) = -i \int_{C_s} d\eta_M, \quad s = 1, \dots, g, \qquad \varphi_j(M) = \begin{bmatrix} \mathrm{Re} \\ {\scriptstyle \mathrm{or}} \\ \mathrm{Im} \end{bmatrix} \int_{F_j} d\eta_M, \quad j = 1, \dots, l,$$

where C_s is the fixed basis in the lattice of odd 1-cycles on the curve M; F_j is a path on M (see Fig. 1) connecting a pair of zeros $(a_j, \pm w(a_j))$ of the differential $d\eta_M$ with the same projection a_j. For a zero a_j lying on the projection of a real oval in M one must consider in the last formula the real part of the integral; in the case of a zero on a coreal oval in M one considers the imaginary part of the integral; for a pair of complex conjugate zeros a_j, $\overline{a}_j$ one takes the real part of one integral and the imaginary part of the other.

From the topological structure of the tree $[\Gamma]$ one can derive the multiplicities of all zeros of $d\eta_M$ and their equality to ramification points of the curve $M\{\Gamma\}$. Hence the image of the coordinate space of the graph Γ lies in one stratum of the moduli space. In this image $\mathcal{A}[\Gamma]$ the above-mentioned coordinate variables on the stratum are integral linear combinations of the heights $H(R)$ and the widths $W(V)$. However, an injective linear map is an embedding, and the injectivity was established in Theorem 3.

The coordinate spaces $\mathcal{A}[\Gamma]$ form a cell decomposition of the moduli space $\mathcal{H}$. We indicate without proof neighbourhood relations existing between the cells. At the boundary of a coordinate space some inequalities in (14), (15) become nonstrict. If the width of a horizontal edge vanishes, then the corresponding edges are contracted into a vertex of the graph. The same occurs in the case when the height of a vertical edge vanishes, however there can be a further transformation to comply with condition (A_2): 

is transformed as $H \to 0$ into the fragment 

if $W_1 < W_2$. In Strebel's terminology this corresponds to the contraction of a strip in the foliation corresponding to $(d\eta_M)^2$. The boundaries of the cells $\mathcal{A}[\Gamma]$ can reach the boundary of the moduli space $\mathcal{H}_g^k$ if the corresponding transformation of the graph $[\Gamma]$ changes its genus. This reflects the fact that moduli spaces of lower genera lie on the boundary of moduli spaces of higher genera.

Lifted to the universal cover of the moduli space, in intersection with a fibre of the period map each $\mathcal{A}[\Gamma]$ produces a cell: in the coordinate space of the graph we must fix several linear combinations of the variables $H(R)$. The partitioning of the moduli space into the coordinate spaces of graphs induces in this way a partitioning of the fibres of the map Π into cells and allows one to study their topology. Unfortunately, calculations of this kind are very labour-consuming even for low genera g.

§ 4. Range of the period map

Proof of Theorem 2. By Lemma 1 the period map commutes with the action of the braid group Br_{g-k+1}, therefore finding the range of Π falls in a natural way into the following two steps:

(1) describe the action of the braid group Br_{g-k+1} on functionals in $(H_1^-(M_0))^*$;

(2) find the Π-image of a representative of each orbit of the covering transformation group in $\widetilde{\mathcal{H}}_g^k$.

(1) The braid group $\mathrm{Br}_{g-k+1} \cong \mathrm{Mod}(\mathbb{H} \setminus \mathbf{e}^0)$ has $g-k$ generators $\beta_1, \dots, \beta_{g-k}$ represented by half-twists along curves encircling a pair of punctures in $\mathbb{H} \setminus \mathbf{e}^0$ (see Fig. 2(b)) [7]. The twist β_s fixes all elements of the distinguished basis except C_{s+k-1} and C_{s+k} which are transformed by the Picard–Lefschetz formula [8]. The corresponding action of the generator on the space of functionals respects all coordinates but two: $\beta_s \cdot (\gamma_{s+k-1}, \gamma_{s+k})^t = (-\gamma_{s+k}, 2\gamma_{s+k} + \gamma_{s+k-1})^t$. In terms of the variable h related to the γ-coordinates by formulae (10) this corresponds to Burau multiplication (9) of braids by vectors.

(2) We supplement each curve M in the moduli space $\mathcal{H}_g^k$ represented by a divisor $\mathbf{e}$ with a special labyrinth Λ, thus lifting it to the universal cover. On the boundary of the domain $\mathbb{H} \setminus \Gamma(M)$ we mark points $v_0, v_1, \dots, v_{g+1}$: two on the real axis to the right and to the left of the interval Γ^0; one point in the projection of each finite real oval and one point at each vertex $V \in \mathbf{e} \cap \mathbb{H}$. We denote the indices of the $g-k+1$ points in the last group by $\mathfrak{i}$. We can use our freedom in the choice of the marked points so that the heights $h_s := H(v_s)$ satisfy conditions (8). Conversely, each point h in the simplex (8) and each admissible index subset $\mathfrak{i}$ can be obtained from the consideration of trees of the form indicated in Fig. 11(a). The first k intervals of the special labyrinth Λ are uniquely defined: these are coreal ovals. The remaining arcs in the labyrinth connect pairs of complex conjugate points v_s, $\overline{v_s}$, $s \in \mathfrak{i}$, are disjoint from the tree $\Gamma(M)$ (except at their end-points, of course) and meet the real axis to the right of the point v_0 (see Fig. 11(b)). To calculate the period map at a point $(\mathbf{e}, \Lambda)$ in the universal cover of the moduli

space one must integrate $d\eta_M$ over the elements of the basis of odd homologies of this point translated from the distinguished point $\widetilde{M}_0 \in \widetilde{H}^k_g$. We know that after the translation of the distinguished basis in $\widetilde{M}_0$ with the use of the natural flat connection we obtain the distinguished basis of the labyrinth Λ. The integrals over its elements can be expressed in terms of the heights h_s. We now perform the corresponding calculations.

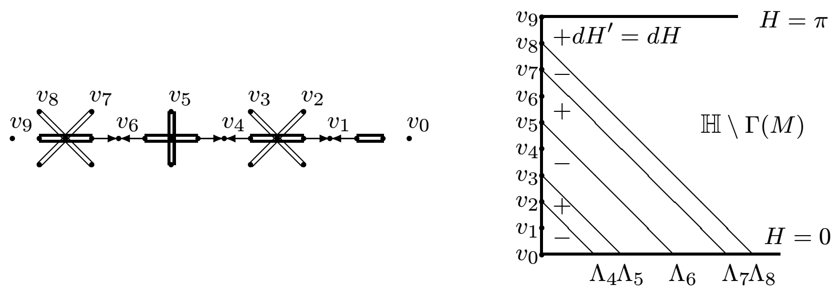


Figure 11. (a) A tree producing $\mathfrak{i} = \{2, 3, 5, 7, 8\}$; (b) the special labyrinth Λ in $\mathbb{H}\setminus\Gamma$

The differential $d\eta_M$ is holomorphic in the simply connected domain $\mathbb{H} \setminus \Lambda$, therefore we can define there a global height function

$$H'(x) := \operatorname{Im} \int_{x_0}^{x} d\eta_M, \qquad x_0 > \Gamma^0;$$

the branch of the differential is normalized by the condition $\operatorname{Res} d\eta_M\big|_\infty = -1$. The same equality defines a global height function $H(x)$ in $\mathbb{H} \setminus \Gamma(M)$. In components of $\mathbb{H} \setminus (\Gamma \cup \Lambda)$ the differentials $dH(x)$ and $dH'(x)$ are equal up to the sign, which changes after crossing Λ or $\Gamma_{\mathfrak{l}}$. In the domain bounded by Λ_s, Λ_{s+1}, and Γ we have $dH'(x) = (-1)^{g+s} dH(x)$.

The value of Π on the cycle C_s, $s = k, \ldots, g$, is four times the increment of the height H' along the right-hand bank of the cut Λ_s – from the real axis to the finite point v_*. The index of this finite point can be conveniently expressed in terms of the right action of braids $\varkappa(\mathfrak{i})$ on the index set $s = 0, 1, \ldots, g+1$ by permutations. For instance, $3 \cdot \varkappa(\mathfrak{i}) = 7$ for the braid in Fig. 3. We finally obtain

$$\gamma_s = 4(-1)^{g+s} h_{(s-k)\cdot\varkappa(\mathfrak{i})}, \qquad s = k, \ldots, g.$$

The value of Π on the cycle C_s, $s = 0, \ldots, k-1$, is equal to twice the increment of the height H' from the point v_* in the component of $\mathbb{R} \setminus \bigcup_{j=0}^{k-1} \Lambda_j$ to the right of Λ_s to the point v_{**} in the component to the left. Expressing the indices of the points v_* and v_{**} in terms of the braid $\varkappa(\mathfrak{i})$ we obtain

$$\gamma_s = 2H'(v_{(g-s+1)\cdot\varkappa(\mathfrak{i})}) - 2H'(v_{(g-s)\cdot\varkappa(\mathfrak{i})}), \qquad s = 0, \ldots, k-1.$$

The heights $H'(v_*)$ in the last formula can be expressed in terms of the known heights h_s. For $j = g-k+1, \ldots, g+1$ we find the smallest index $m = m(j)$ in the set $0, \ldots, g-k$ such that $j \cdot \varkappa(\mathfrak{i}) < m \cdot \varkappa(\mathfrak{i})$; for instance, for the braid $\varkappa(\mathfrak{i})$

in Fig. 3 we have $m(7) = 2$. If there exists no such index m, then $H'(v_{j\cdot\varkappa(\mathrm{i})}) = h_{j\cdot\varkappa(\mathrm{i})}$, otherwise

$$\begin{aligned} H'(v_{j\cdot\varkappa(\mathrm{i})}) &= \bigl(H'(v_{j\cdot\varkappa(\mathrm{i})}) - H'(v_{m\cdot\varkappa(\mathrm{i})})\bigr) + \bigl(H'(v_{m\cdot\varkappa(\mathrm{i})}) - H'(v_{(m+1)\cdot\varkappa(\mathrm{i})})\bigr) \\ &\quad + \dots + \bigl(H'(v_{(g-k-1)\cdot\varkappa(\mathrm{i})}) - H'(v_{(g-k)\cdot\varkappa(\mathrm{i})})\bigr) + H'(v_{(g-k)\cdot\varkappa(\mathrm{i})}) \\ &= 2h_{(g-k)\cdot\varkappa(\mathrm{i})} - 2h_{(g-k-1)\cdot\varkappa(\mathrm{i})} \\ &\quad + 2h_{(g-k-2)\cdot\varkappa(\mathrm{i})} - \dots \pm 2h_{m\cdot\varkappa(\mathrm{i})} \mp h_{s\cdot\varkappa(\mathrm{i})}. \end{aligned}$$

The Burau representation now allows one to write the results of our calculations in a compact form.

The above method of finding the range of the period map produces a slightly different answer for moduli spaces with $k = 0$. In the space $\mathbb{R}^{g+1}$ with variables $(h_0, \dots, h_g)^t$ we define an open g-dimensional polyhedron Δ_g^0 as the section of the interior of the simplex $\{0 < h_0 < h_1 < \dots < h_g < \pi\}$ by a hyperplane $\{h_g - h_{g-1} + \dots \pm h_1 \mp h_0 = \pi/2\}$.

Theorem 5. *The set $\Pi(\widetilde{\mathcal{H}}_g^0)$ is the image under the linear map $\gamma_s := 4(-1)^{g+s}h_s$, $s = 0, 1, \dots, g$ of the union of the open polyhedra $\beta \cdot \Delta_g^0 \ni (h_0, h_1, \dots, h_g)^t$ taken over various braids β on $g+1$ strands.*

Theorems 2 and 5 give an exhaustive answer to the first question in the introduction. Our techniques give an affirmative answer to the second question: some fibres of the period map (the inverse images of points in $(H_1^-(M_0, \mathbb{R}))^*$) are fixed by non-trivial covering transformations of the universal cover. In fact, let $h \in \varkappa(\mathrm{i}) \cdot \Delta_g$ be a point such that its coordinate h_s is the arithmetic mean of the neighbouring coordinates h_{s-1} and h_{s+1}, $s \in \{1, 2, \dots, g-k-1\}$. If $s = 1$, then, for instance, the curve M in the space $\mathcal{H}_3^1$ with graph depicted in Fig. 12 has this property. One can verify that $\beta_s^{-1}\beta_{s+1}\beta_s \cdot h = h$ in this case. The topology of the projection of a fibre onto the moduli spaces $\mathcal{H}_g^k$ can be distinct from the topology of the fibre itself.

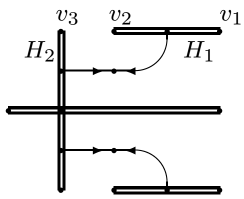


Figure 12. The graph $\Gamma(M)$ of a curve corresponding for $H_1 = H_2$ to fixed points of the braid group action in the range of Π

§ 5. Classification of extremal polynomials

Another application of our graph techniques is a geometric representation and a classification of g-extremal polynomials by means of the graphs of the curves assigned to them. The tradition of the use of graphs in the description of polynomials and other algebraic functions goes back to Hurwitz's well-known paper of 1891. One connects the critical points of the polynomial by rays of a 'star' (a tree, but with hanging edges) the inverse image of which is a tree of a special form, dubbed a 'cactus' [12]. Adapting the 'star' to our case we arrive at the graph of the associated curve. In the plane of the values of the polynomial $P(x)$ we draw the interval $[-1,1]$ red and connect each critical value of $P(x)$ with $[-1,1]$ by an interval from the foliation of confocal hyperbolae with foci at ± 1, which we draw black. We lift this figure to the plane of the independent variable x of the polynomial and wipe off the hanging edges one after another. The result is just the graph Γ; its red part will be 'vertical' and black part 'horizontal'. The graphs associated with polynomials satisfy (in addition to (A_1)–(A_5)) an additional constraint on the heights of the vertical edges.

Theorem 6. *A weighted graph $\{\Gamma\}$ corresponds to a polynomial of degree n if and only if for each vertex $V \in \Gamma^0 \cup \Gamma^+$ of odd order $\operatorname{ord}(V)$ (a branch point) there exists a corresponding vertex v in the boundary of $\mathbb{H} \setminus \Gamma$ of height $H(v) \in \pi\mathbb{Z}/n$.*

Proof. (1) If a curve M corresponds to a polynomial of degree n, then (see [2]) Abel's equations (6) hold and

$$2\pi i\mathbb{Z}/n \ni \int_{\bar{v}}^{v} d\eta_M = 2iH(v);$$

here the integration path bypasses the graph $\Gamma(M)$ on the right.

(2) Conversely, consider the special basis in the lattice of odd 1-cycles constructed in the proof of Theorem 2. It is clear from formula (10) that if $h_s \in \pi\mathbb{Z}/n$, then the periods of $d\eta_M$ over the odd cycles $2C_0, 2C_1, \dots, 2C_{k-1}; C_k, \dots, C_g$ belong to $4\pi i\mathbb{Z}/n$. Hence [2] the curve M corresponds to a polynomial of degree n.

Under the assumptions of the last theorem the height of each vertex on the boundary of the cut half-plane corresponding to a ramification point of the curve belongs to $\pi\mathbb{Z}/n$.

Lemma 6. *Let M be a curve constructed from a polynomial of degree n, and for a point in an edge R of the graph $\Gamma(M)$ of this curve let v, v' be the corresponding points in the boundary of the half-plane cut along the graph. Then $H(v) - H(v')$ lies in $2\pi\mathbb{Z}/n$ if $R \subset \Gamma_{-}(M)$; $H(v) + H(v') \in 2\pi\mathbb{Z}/n$ if $R \subset \Gamma_{|}(M)$.*

Proof. Let $R \subset \Gamma_{-}$. We draw an arc C from v' to v disjoint from the graph. This arc encircles an even number of branch points of the curve and therefore defines an element of the $\mathbb{Z}$-homology on M. We project C onto the odd 1-cycles; then by Abel's equations (6) we obtain

$$H(v) - H(v') = -i \int_C d\eta_M \in 2\pi\mathbb{Z}/n.$$

On the other hand, if $R \subset \Gamma_{|}$, then we choose a vertex v'' corresponding to a branch point, with height $H(v'') \in \pi\mathbb{Z}/n$. We draw two arcs C and C' disjoint from the graph: from v'' to v and from v'' to v', respectively. We represent M as two sheets $\mathbb{C} \setminus \Gamma_{|}$ glued crosswise along the cuts. Let J be the transposition of the sheets (the hyperelliptic involution); then $C - JC'$ defines an integral 1-cycle on M and therefore

$$H(v) + H(v') - 2H(v'') = -i \int_{C - JC'} d\eta_M \in 2\pi\mathbb{Z}/n.$$

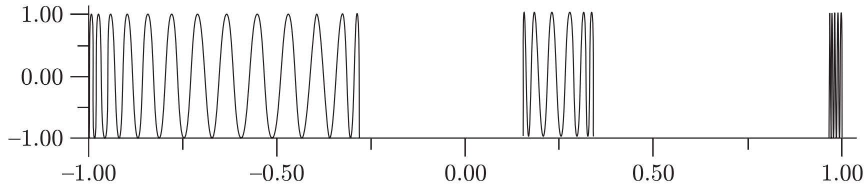


Figure 13. The polynomial $P_{50}(x)$ with graph $\frac{27}{50}\pi$ $\frac{12}{50}\pi$ $\frac{11}{50}\pi$

It is a characteristic feature of the g-extremal polynomials used as substitutions in the solution of least deviation problems that their non-exceptional critical points lie on the real axis. Accordingly, the classifying graph $\{\Gamma\}$ of the normalized polynomial must be concentrated on $\mathbb{R}$ in some sense: for instance, $\sum_{R\subset\Gamma^0} H(R)\approx\pi$. We claim that oscillatory behaviour of the polynomial on the real axis, that is, its qualitative graph is defined by a neighbourhood of the graph Γ^0. Assume that a polynomial $P_n(x)$ with positive leading coefficient corresponds to a graph Γ in the plane. At a point v on the boundary of the upper half-plane cut along the graph the polynomial takes the following value (5):

$$P_n(v)=\cos ni(W(v)+iH(v))=\cos nH(v)\cosh nW(v)+i\sin nH(v)\sinh nW(v), \tag{17}$$

which by Lemma 6 does not change when crossing cuts. If $v\in\mathbb{R}$, then the imaginary part of this value vanishes since $W(v):=0$ on $\Gamma^0_{|}$, $H(v)\in\pi\mathbb{Z}/n$ on Γ^0_{-}, and $H(v)\in\{0,\pi\}$ on $\mathbb{R}\backslash\Gamma^0$. The multiplicity of the value (17) is by the same formula (5) equal to

$$\operatorname{mult} P_n(v)=(1+\operatorname{ord}(v)/2)\cdot\begin{cases}1, & H(v)-iW(v)\notin\pi\mathbb{Z}/n,\\ 2, & H(v)-iW(v)\in\pi\mathbb{Z}/n.\end{cases}$$

In particular, each edge $(v^-,v^+)\subset\Gamma^0_{|}$ contains $|[-nH(v^-)/\pi]+[nH(v^+)/\pi]+1|$ simple critical points of $P_n(x)$, whose values alternate between 1 and −1. We have thus found all the critical points of the polynomial on the real axis and the corresponding critical values, with their multiplicities. This is sufficient for plotting a qualitative graph of P_n. The extremal polynomials can also be calculated using computer software (see Fig. 13), in the spirit of [3] and [4].

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