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Representation of moduli spaces of curves and calculation of extremal polynomials

A. B. Bogatyrëv

Abstract. The classical Chebyshev and Zolotarëv polynomials are the first ranks of the hierarchy of extremal polynomials, which are typical solutions of problems on the conditional minimization of the uniform norm over a space of polynomials. In the general case such polynomials are connected with hyperelliptic curves the genus of which labels the ranks of the hierarchy. Representations of the moduli spaces of such curves are considered in this paper with applications to the calculation of extremal polynomials. Uniformizing curves by special Schottky groups one obtains effectively computable parametric expressions for extremal polynomials in terms of linear series of Poincaré.

Bibliography: 12 titles

§ 1. Introduction

150 years ago, Chebyshev and his school started the investigation of problems of the conditional minimization, over the space of real polynomials $P(x)$, of their *deviation* $\|P\|_E := \max_{x\in E}|P(x)|$, where E is a compact subset of the real axis. Typical constraints in such a problem are an upper bound on the degree $n = \deg P$ of the polynomial and fixed values of its derivatives $P^{(m)}(x)$, $m = 0, 1, 2, \dots$, at certain fixed points $x \in \mathbb{C}$.

Nowadays, interest in least deviation problems relates, for instance, to the optimization of numerical methods and signal processing. Iterative methods of conditional minimization (see the references in [1]) are very labour-consuming for high degrees n of the solution. The classical approach, when the solution is normally given by an explicit formula, is free from this deficiency. The first least deviation problems were solved in the form of parametric expressions (Chebyshev, 1853, and Zolotarëv, 1868 [2]):

$$T_n(u) := \cos(nu), \qquad x(u) := \cos(u), \quad u \in \mathbb{C}, \tag{1}$$

$$Z_n(u) := \frac{1}{2}\left\{\left[\frac{H(a+u)}{H(a-u)}\right]^n + \left[\frac{H(a-u)}{H(a+u)}\right]^n\right\}, \quad x(u) := \frac{\operatorname{sn}^2(u)+\operatorname{sn}^2(a)}{\operatorname{sn}^2(u)-\operatorname{sn}^2(a)}, \quad u \in \mathbb{C}, \tag{2}$$

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where $H(\,\cdot\,)$ is the elliptic theta function in the (now outdated) Jacobi notation, $\operatorname{sn}(\,\cdot\,)$ is the elliptic sine function of the same modulus $k \in (0,1)$, $a := mK(k)/n$, $m = 1, 2, \dots, n-1$, is the phase shift, and $K(k)$ is the full elliptic integral of modulus k. The expressions T_n and Z_n are x-polynomials of degree n satisfying the following definition. We say that a real polynomial is (*normalized*) *extremal* if all its critical points, except a small number g of them, are simple and correspond to the values ± 1. For the Chebyshev polynomials T_n one has $g = 0$ and for the Zolotarëv polynomials Z_n, $g = 1$. The theory of general extremal polynomials for $g = 0, 1, 2, \dots$ has been developed in [1]. We now present the requisite information on the representation of such polynomials.

A construction going back to Chebyshev associates with a real polynomial $P_n(x)$ the real hyperelliptic curve

$$M = M(\mathbf{e}) := \left\{ (x, w) \in \mathbb{C}^2 : w^2 = \prod_{s=1}^{2g+2} (x - e_s) \right\}, \qquad \mathbf{e} := \{e_s\}_{s=1}^{2g+2}, \tag{3}$$

with branching divisor $\mathbf{e}$ equal to the odd-order zeros of the polynomial $P_n^2(x) - 1$. If $P_n(x)$ is a normalized extremal polynomial, then the genus g of M is not large; it is equal to the properly counted number of exceptional critical points of the polynomial. The polynomial of degree n can be recovered from the associated curve (3) up to a sign by an explicit formula generalizing (1) and (2):

$$P_n(x) = \pm \cos\left(ni \int_{(e,0)}^{(x,w)} d\eta_M \right), \qquad x \in \mathbb{C}, \quad (x, w) \in M, \tag{4}$$

where $d\eta_M := \prod_{s=1}^{g} (x - c_s) \dfrac{dx}{w}$ is an Abelian differential with purely imaginary periods. The curve (3) associated with a polynomial of degree n satisfies the system of Abel's equations

$$-i \int_{C_s^-} d\eta_M = 2\pi \frac{m_s}{n}, \qquad s = 0, 1, \dots, g, \tag{5}$$

where $\{C_s^-\}_{s=0}^{g}$ is a basis in the lattice of integral 1-cycles on the curve M changing sign after the anticonformal involution $\overline{J}(x, w) := (\overline{x}, \overline{w})$, and the m_s are integers from the domain described in [3].

The aim of the present paper is twofold:

(A) *to solve effectively Abel's equations* (5) *in the moduli space of the curves* M;

(B) *to calculate effectively by formula* (4) *extremal polynomials and their derivatives of various orders so that they comply with the constraints of the least deviation problem.*

In the special case of Chebyshev polynomials on several intervals this aim was attained in [4] by means of the uniformization of the curves M in question by Schottky groups. The extension of these techniques to the general case comes up against a more complicated topology of the moduli space of the curves (3). A component of this space is homeomorphic to the product of a cell by the configuration space of a (half-)plane and its fundamental group is an Artin braid group. Our problems (A) and (B) require an organization of iterative computations on moduli spaces; to this end we perform an analytic uniformization of these spaces.

§ 2. Representations of moduli spaces

Real hyperelliptic curves (3) associated with polynomials are distinguished by means of Abel's equations (5) defined globally on the universal cover of the curve moduli space. It can be useful to look at an object of investigation from various standpoints, therefore we present four definitions of this space and demonstrate their equivalence. In the standard fashion the *universal cover of the moduli space* is defined as the set of branching divisors $\mathbf{e}$ of the prescribed type together with the history of their motion starting from a fixed divisor $\mathbf{e}^0$. By considering a divisor as moving in a viscous medium and carrying with it particles of this medium we arrive at the *Teichmüller space* of a punctured disc with distinguished boundary points, a flexible technique, which reveals connections that exist between views from various standpoints. The *deformation spaces* of special Kleinian groups bring forth global coordinates in the space under study and allow an effective construction of analytic objects. *Labyrinth spaces*, the most geometric of them all, enable one to calculate [3] the range of the period map defined by the left-hand sides of Abel's equations.

2.1. Four definitions. We fix the topological invariants of a real curve (3): its genus $g = 0, 1, 2, \dots$ and the number of coreal ovals $k = 0, 1, \dots, g+1$. A symmetric divisor $\mathbf{e}$ of type (g, k) is an unordered set of distinct points $e_1, \dots, e_{2g+2}$ consisting of $2k$ real points and $g-k+1$ pairs of complex conjugate ones. On such sets we have a free action of the group $\mathfrak{A}_1^+$ of orientation-preserving affine motions of the real axis: $\mathbf{e} = \{e_s\}_{s=1}^{2g+2} \to A\mathbf{e} + B = \{Ae_s + B\}_{s=1}^{2g+2}$, $A > 0$, $B \in \mathbb{R}$. We shall call the orbits of this action the *moduli space* $\mathcal{H}_g^k$. Points in the moduli space correspond to conformal classes of real hyperelliptic curves (3) with fixed invariants g, k and distinguished point ∞_+ on the oriented real oval. The space $\mathcal{H}_g^k$ has the natural structure of a real $2g$-manifold. For the introduction of local coordinates in the neighbourhood of a fixed divisor $\mathbf{e}^0$ we number the point in the set $\mathbf{e}^0 = \{e_s\}_{s=1}^{2g+2}$ and fix a pair of complex conjugate or a pair of real points e_{2g+1}, e_{2g+2}. For local coordinate variables we take the quantities $\operatorname{Re} e_s$ and $\operatorname{Im} e_s$ for the points e_s in the open upper half-plane $\mathbb{H}$ and $\operatorname{Re} e_s$ for the real points e_s, $s = 1, 2, \dots, 2g$.

Lemma 1 [1]. *The fundamental group of the moduli space $\pi_1(\mathcal{H}_g^k)$ is isomorphic to the group Br_{g-k+1} of Artin braids on $g-k+1$ strings.*

The space of classes of homotopically equivalent paths in $\mathcal{H}_g^k$ starting at the distinguished point $\mathbf{e}^0$ in the moduli space is called the *universal cover* $\widetilde{\mathcal{H}}_g^k(\mathbf{e}^0)$. It has three representations described below.

2.1.1. Teichmüller space. *Quasiconformal* homeomorphisms of the upper half-plane $\mathbb{H}$ with fixed infinity form a group $QC(\mathbb{H})$ with respect to taking composites. Each map $f \in QC(\mathbb{H})$ can be extended to a quasiconformal homeomorphism of $\mathbb{C}$ by means of the reflection relative to the real axis. The motions f stabilizing the fixed branching divisor $\mathbf{e}^0$ (but possibly rearranging points in it) make up a subgroup $QC(\mathbb{H}, \mathbf{e}^0)$. The motions f joined to id by a homotopy of the punctured sphere $\mathbb{CP}_1 \setminus \mathbf{e}^0$ stabilizing infinity make up a subgroup $QC^0(\mathbb{H}, \mathbf{e}^0)$ of $QC(\mathbb{H}, \mathbf{e}^0)$. It acts on $QC(\mathbb{H})$ by right multiplications, while the affine group $\mathfrak{A}_1^+$ acts by left multiplications. These actions commute and the well-defined two-sided

quotient $\mathcal{T}_g^k(\mathbf{e}^0) := \mathfrak{A}_1^+ \setminus QC(\mathbb{H})/QC^0(\mathbb{H},\mathbf{e}^0)$ is called the *Teichmüller space.*[1] By the Teichmüller distance between classes $[f],[h] \in \mathcal{T}_g^k(\mathbf{e}^0)$ one means the minimum over all representatives $f_1 \in [f]$, $h_1 \in [h]$ of the logarithm of the dilatation of the quasiconformal map $f_1 h_1^{-1}$.

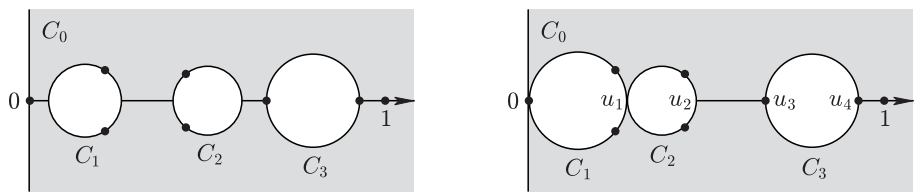
The *modular group* $\mathrm{Mod}(\mathbf{e}^0) := QC(\mathbb{H},\mathbf{e}^0)/QC^0(\mathbb{H},\mathbf{e}^0)$ acts by right multiplications on the Teichmüller space and the corresponding automorphisms are isometries. One's choice of a distinguished divisor $\mathbf{e}^0$ in the definition of the Teichmüller space is not essential: a motion $h \in QC(\mathbb{H})$ gives rise to an isometry of $\mathcal{T}_g^k(\mathbf{e}^0)$ onto $\mathcal{T}_g^k(h\mathbf{e}^0)$ by the formula $f \to fh^{-1}$. Obviously, $QC(\mathbb{H},h\mathbf{e}^0) = hQC(\mathbb{H},\mathbf{e}^0)h^{-1}$ and $QC^0(\mathbb{H},h\mathbf{e}^0) = hQC^0(\mathbb{H},\mathbf{e}^0)h^{-1}$, therefore the modular groups $\mathrm{Mod}(\mathbf{e}^0)$ and $\mathrm{Mod}(h\mathbf{e}^0)$ are isomorphic and our isometry $\mathcal{T}_g^k(\mathbf{e}^0) \to \mathcal{T}_g^k(h\mathbf{e}^0)$ commutes with the action of the modular groups in each space.

Assigning to a motion $f \in QC(\mathbb{H})$ the branching divisor $\mathbf{e} := f(\mathbf{e}^0)$ we obtain a projection of the Teichmüller space onto the moduli space. The fibres of this projection are orbits of the modular group. We shall demonstrate that this projection coincides with the universal cover.

2.1.2. Deformation space of the group. We partition the index set $\{0,1,\dots,g\}$ into two subsets: a $(g-k+1)$-element one $\mathfrak{i}$ and its k-element complement $\mathfrak{i}'$. The *deformation space* $\mathcal{G}_g^k(\mathfrak{i})$ is formed by ordered sets $\{G_s\}_{s=0}^g$ of linear fractional rotations of the second order with real fixed points $c_s \pm r_s$ for $s \in \mathfrak{i}'$ or with complex conjugate fixed points $c_s \pm ir_s$ for $s \in \mathfrak{i}$:

$$G_0 u := \begin{cases} -u, & 0 \in \mathfrak{i}', \\ -\dfrac{1}{u}, & 0 \in \mathfrak{i}, \end{cases} \qquad G_s u := \begin{cases} c_s + \dfrac{r_s^2}{u - c_s}, & s \in \mathfrak{i}', \\ c_s - \dfrac{r_s^2}{u - c_s}, & s \in \mathfrak{i}, \end{cases} \qquad s = 1,2,\dots,g. \quad (6)$$

The real parameters c_s and r_s (the moduli) are selected so that the following geometric condition holds. *There exist g disjoint subintervals of $(0,u_\infty)$ numbered in increasing order such that the circles $C_1,C_2,\dots,C_g$ with diameters on these intervals pass through the fixed points of the corresponding motions $G_1,G_2,\dots,G_g$* (see Fig. 1(a)). The distinguished point u_∞ is $+\infty$ if $0 \in \mathfrak{i}$, and $u_\infty := 1$ for $0 \in \mathfrak{i}'$.



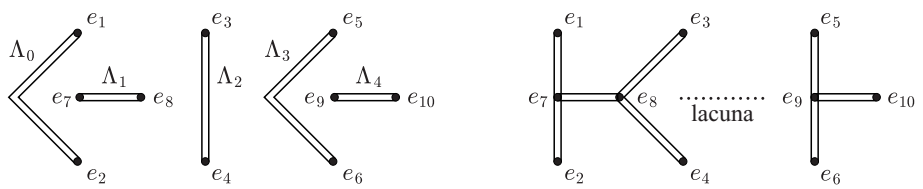
(a) The circles $C_1,C_2,\dots,C_g$ for $\mathfrak{i} = \{1,2\}$ (b) The limiting position of the circles

Figure 1

[1]This is a modification of the standard definition [5] of the Teichmuller space of a disc with $g-k+1$ punctures and $2k+1$ distinguished points at the boundary.

If this condition is fulfilled, then the y-axis C_0 and the circles $C_1, C_2, \dots, C_g$ bound the fundamental domain $\mathcal{R}$ of the Kleinian group $\mathfrak{G}$ generated by the rotations $G_0, G_1, \dots, G_g$. By Klein's combination theorem [6] the group $\mathfrak{G}$ is the free product of $g+1$ second-order groups. The hyperbolic motions $\{S_l := G_l G_0\}_{l=1}^{g}$ generate the Schottky group $\mathfrak{S}$, $|\mathfrak{G} : \mathfrak{S}| = 2$. These two groups have a common domain of discontinuity $\mathcal{D}$ and a limit set lying on the real axis. The linear measure of the limit set is zero since the group $\mathfrak{S}$ satisfies the following Schottky criterion [7]: the *fundamental domain* $\mathcal{R}(\mathfrak{S})$ (= the exterior of the $2g$ circles $G_0C_g, \dots, G_0C_1$; $C_1, \dots, C_g$) *can be partitioned into triply connected domains* (= *pants*) *by additional circles.* This is crucial for our aims because the Poincaré linear theta series will converge absolutely and uniformly on compact subsets of the domain of discontinuity of $\mathfrak{S}$.

The orbit manifold of the group $\mathfrak{G}$ is the Riemann sphere with natural reflection $\overline{J}u := \overline{u}$. The quotient manifold $\mathcal{D}/\mathfrak{S}$ is a compact algebraic curve M_c of genus g with hyperelliptic involution $Ju := G_0 u$ and anticonformal involution $\overline{J}$. A holomorphic projection $x(u)\colon \mathcal{D} \to \mathbb{CP}_1 \cong \mathcal{D}/\mathfrak{G}$ with a pole at the distinguished point u_∞ that respects complex conjugation and preserves the orientation of the real axis in the neighbourhood of $u = u_\infty$ is defined uniquely up to motions in $\mathfrak{A}_1^+$. We say that such a branched cover $x(u)$ is *compatible* with the group $\mathfrak{G}$. Assigning to the Kleinian group the branching points of $x(u)$ (= the projections onto the sphere of the fixed points of the rotations $\{G_s\}_{s=0}^{g}$) we define a map from the deformation space $\mathcal{G}_g^k$ into the moduli space $\mathcal{H}_g^k$. We show in what follows that this is the universal cover and the modular group acts on the deformation space preserving the Kleinian group $\mathfrak{G}$ (up to conjugation), but changing the system of its generators.



(a) The labyrinth $(\mathbf{e}, \Lambda)$ for $g = 4, k = 2, \mathfrak{i} = \{0, 2, 3\}$ (b) A modification of a labyrinth

Figure 2

2.1.3. Labyrinth space. By a *labyrinth* $(\mathbf{e}, \Lambda)$ of type $(g, k, \mathfrak{i})$ we shall mean a symmetric divisor $\mathbf{e}$ of type (g, k) supplemented with a system of disjoint cuts $\Lambda := (\Lambda_0, \Lambda_1, \dots, \Lambda_g)$ connecting pairwise points in $\mathbf{e}$. The first group of cuts are the projections of the k coreal ovals of the curve $M(\mathbf{e})$, that is, the components of the set $\{x \in \mathbb{R} : w^2(x) < 0\}$. The second group is a system of smooth simple arcs connecting complex conjugate points $\mathbf{e}$ that are invariant under the reflection relative to $\mathbb{R}$. The intersections with the real axis define an ordering of the cuts, which we number from 0 to g from left to right (see Fig. 2(a)). The indices of the cuts in the second group form the set $\mathfrak{i}$.

Two labyrinths $(\mathbf{e}, \Lambda)$ and $(\mathbf{e}', \Lambda')$ are considered equivalent if there exists a motion in $\mathfrak{A}_1^+$ taking $\mathbf{e}$ to $\mathbf{e}'$ and the paths Λ into paths continuously deformable into

Λ' so that the deformed paths and the point set $\mathbf{e}'$ form a labyrinth at each instant. We call the quotient of the set of labyrinths of type $(g, k, \mathfrak{i})$ by this equivalence relation the *labyrinth space* $\mathcal{L}_g^k(\mathfrak{i})$. Wiping away the cuts one obtains a natural projection $\mathcal{L}_g^k(\mathfrak{i}) \to \mathcal{H}_g^k$.

The idea behind the introduction of labyrinths is as follows: the cuts of a labyrinth transform the punctured half-plane $\mathbb{H} \setminus \mathbf{e}$ into a simply connected set. On the one hand this fixes generators of the free group $\pi_1(\mathbb{H} \setminus \mathbf{e})$ and on the other, it allows one to trace the dynamics of the punctures.

2.2. Auxiliary results. The proof of the equivalence of the four spaces $\widetilde{\mathcal{H}}_g^k(\mathbf{e}^0)$, $\mathcal{T}_g^k(\mathbf{e}^0)$, $\mathcal{G}_g^k(\mathfrak{i})$, $\mathcal{L}_g^k(\mathfrak{i})$ introduced above is based on their properties to be established in this subsection.

2.2.1. Topology of the deformation space. The moduli $c_s, r_s > 0$, $s=1,\dots,g$, form a global system of coordinates in the deformation space $\mathcal{G}_g^k(\mathfrak{i})$ and allow one to identify it with a subdomain of $\mathbb{R}^{2g}$.

Lemma 2. *The space $\mathcal{G}_g^k(\mathfrak{i})$ is the cell described by the system of inequalities*

$$r_s > 0, \qquad s = 1, 2, \dots, g, \tag{7}$$

$$\begin{aligned} c_s + r_s < G_{s+1}(c_s + r_s) < G_{s+2}G_{s+1}(c_s + r_s) < G_{s+3}G_{s+2}G_{s+1}(c_s + r_s) \\ < \dots < G_{s'-1}G_{s'-2}\cdots G_{s+2}G_{s+1}(c_s + r_s) < c_{s'} - r_{s'}, \end{aligned} \tag{8}$$

where the indices s and s' in the last chain of inequalities answer one of the following four descriptions:

(1) *s and s', $s < s'$, are successive indices in the set $\mathfrak{i}' \setminus \{0\}$;*
(2) *$s = 0$ and s' is the smallest index in $\mathfrak{i}' \setminus \{0\}$; here one sets $c_0 + r_0 := 0$;*
(3) *s is the greatest index in $\mathfrak{i}' \setminus \{0\}$ and $s' = g + 1$, here one sets $c_{s'} - r_{s'} := u_\infty$;*
(4) *$s = 0$ and $s' = g + 1$ if the set $\mathfrak{i}' \setminus \{0\}$ is empty.*

Proof. The rotation G_s, $s \in \mathfrak{i}'$, has real fixed points and therefore the corresponding circle C_s is uniquely defined. On the other hand, one can move the real diameter of the circle C_s, $s \in \mathfrak{i}$. We move all such diameters to the extreme right position (see Fig. 1(b)). The system of inequalities (8) describes the ordering of the end-points of the resulting diameters in the interval $(0, u_\infty)$. One obtains a diameter configuration, which can be uniquely recovered from their end-points, ranging over the cell $\{0 < u_1 < u_2 < \dots < u_\alpha < u_\infty\}$ of dimension $\alpha := 2\#\{\mathfrak{i}'\setminus\{0\}\} + \#\{\mathfrak{i}\setminus\{0\}\}$, and the index set $\mathfrak{i}$. Prescribing a direction from the centre of the sth shifted diameter to the fixed point of the rotation G_s in the upper half-plane, $s \in \mathfrak{i} \setminus \{0\}$, one obtains a point in the cell $(0, \pi)^\beta$, $\beta := \#\{\mathfrak{i} \setminus \{0\}\}$. We have thus constructed a map of the space $\mathcal{G}_g^k(\mathfrak{i})$ onto a cell of dimension $\alpha + \beta = 2g$ which is continuous and one-to-one.

2.2.2. The group of the branched cover $\boldsymbol{x(u)}$. Each labyrinth $(\mathbf{e}, \Lambda)$ defines a representation χ_Λ from the fundamental group of the punctured sphere $\pi_1(\mathbb{CP}_1 \setminus \mathbf{e}, \infty)$ into an abstract group $\mathfrak{G}$ equal to the free product of $g + 1$

groups of the second order with generators $G_0, G_1, \ldots, G_g$. One assigns to a loop ρ intersecting transversally the successive cuts $\Lambda_{s_1}, \Lambda_{s_2}, \ldots, \Lambda_{s_l}$ the element of this group

$$\chi_\Lambda[\rho] := G_{s_1} G_{s_2} \cdots G_{s_l}.$$

A point $\{G_s\}_{s=0}^g$ in the deformation space generates a labyrinth $(\mathbf{e}, \Lambda)$ in accordance with the following rule: $\mathbf{e} := \{x(\operatorname{fix} G_s)\}_{s=0}^g$ is the set of projections of the fixed points of the generators; $\Lambda := (xC_0, xC_1, \ldots, xC_g)$ is the projection of the boundary of the fundamental domain. For $0 \in \mathfrak{i}$ one must shift C_0 away from a pole of $x(u)$, replacing it by the circles $C_\varepsilon := \{u : |\varepsilon u + 1|^2 = \varepsilon^2 + 1\}$ with small $\varepsilon > 0$. The kernel of the corresponding representation χ_Λ is the group of the cover $x(u)$ ramified over $\mathbf{e}$ and related to the element in question of the deformation space. The cover group can be proved to be completely determined by the branching divisor $\mathbf{e}$.

Lemma 3. *The kernel of the representation* $\chi_\Lambda \colon \pi_1(\mathbb{CP}_1 \setminus \mathbf{e}, \infty) \to \mathfrak{G}$ *is independent of the labyrinth* Λ.

Proof. We shall show that $\ker \chi_\Lambda$ is equal to the normal subgroup of $\pi_1(\mathbb{CP}_1 \setminus \mathbf{e}, \infty)$ generated by all elements of the following two kinds:

(a) a lasso making two rounds about the punctures and
(b) a loop λ with mirror symmetry $[\lambda\overline{\lambda}] = 1$ that is disjoint from the cuts Λ_i, $i \in \mathfrak{i}'$.

The above-described subgroup is independent of one's choice of the labyrinth (each labyrinth contains the projections of the coreal ovals of the curve $M(\mathbf{e})$) and obviously lies in the kernel of χ_Λ. We shall now demonstrate the reverse inclusion in the case when the curve contains at least one coreal oval. The case $k = 0$ will require obvious changes in the argument.

A cell decomposition of the Riemann sphere with $2g+2$ vertices $\mathbf{e}$, $2g+1$ oriented edges R, and one 2-cell gives us a system of free generators of the group $\pi_1(\mathbb{CP}_1 \setminus \mathbf{e})$: *one associates with each edge R the class of the loop ρ intersecting only this edge from left to right.* Such a cell decomposition can be constructed from the labyrinth $(\mathbf{e}, \Lambda)$. The intervals Λ_i, $i \in \mathfrak{i}'$, give us k edges R, the *lacunae* between them (= the projections of finite real ovals) give a further $k-1$ edges. The lacking $2(g-k+1)$ edges can be obtained by a modification of the remaining arcs of the labyrinth Λ_i, $i \in \mathfrak{i}$, in the neighbourhood of real ovals of the curve M after which they pass through the punctures on the real axis (see Fig 2(b)).

On the generators related to the edges R the representation χ_Λ acts as follows:

$$\chi_\Lambda[\rho] = \begin{cases} G_s & \text{if } R \text{ is a (modified) cut } \Lambda_s, \\ 1 & \text{if } R \text{ is a lacuna.} \end{cases}$$

Since $\mathfrak{G}$ is the free product of groups of rank 2, the kernel χ_Λ is the normal subgroup generated by all possible elements $[\gamma]$, $[\rho]^2$, $[\rho\overline{\rho}]$, where the $[\gamma]$ correspond to the $k-1$ lacunae and the $[\rho]$ correspond to the other $2g-k+2$ edges R. An exhaustive search demonstrates that all these elements generating $\ker \chi_\Lambda$ belong to the above-described subgroup.

Naturally embedded in the fundamental group of the punctured sphere is the group of the punctured upper hyperplane $\pi_1(\mathbb{H} \setminus \mathbf{e}, \infty)$. As in Lemma 3, the cuts in

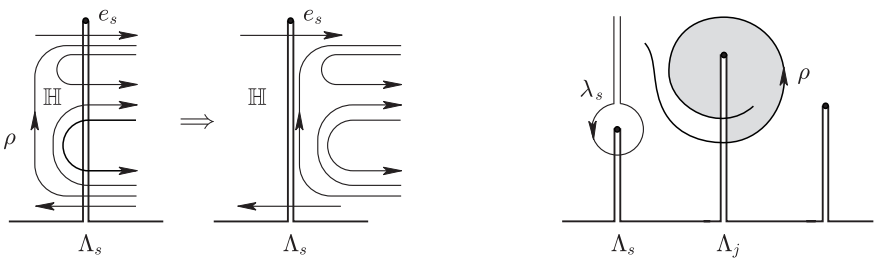
a labyrinth present a system of free generators coding the elements of that group: a loop ρ in the upper hyperplane intersecting transversally the cuts $\Lambda_{s_1}, \Lambda_{s_2}, \dots, \Lambda_{s_l}$ one after another can be expanded in the generators λ_s, $s \in \mathfrak{i}$, intersecting only the cut Λ_s from left to right:

$$[\rho] = [\lambda_{s_1}]^{\varepsilon_1}[\lambda_{s_2}]^{\varepsilon_2} \cdots [\lambda_{s_l}]^{\varepsilon_l}, \tag{9}$$

where $\varepsilon_j = \pm 1$ depending on the orientation of the local intersection of ρ with the cut Λ_{s_j}.

Lemma 4. *Let $\rho \subset \mathbb{H} \setminus \mathbf{e}$ be a loop without self-intersections and with initial point at ∞. Then the irreducible factorization of $[\rho]$ in the generators* (9) *has no equal letters following one another.*

Proof. This is a result of discrete mathematics based on the idea of continuity. The factorization depends only on the homotopy class of the loop ρ, therefore we shall assume without loss of generality that ρ intersects the Λ_s transversally and at finitely many points. Making finitely many transformations of Fig. 3(a) we replace ρ by a homotopic loop without self-intersections with irreducible factorization (9). If this representation contains two successive symbols $[\lambda_j]$, then, up to orientation, we are in the situation of Fig. 3(b). The point going along ρ must return to infinity, but it cannot leave the shaded domain bounded by the loop itself and a piece of the cut Λ_j: otherwise the loop self-intersects or its factorization is reducible.



(a) The elimination of cancellations (b) The infinite spiral ρ and the generator λ_s

Figure 3

2.2.3. Modular group action on the group $\mathfrak{G}$. The natural action of homeomorphisms $f \in QC(\mathbb{H}, \mathbf{e})$ on the fundamental group of the punctured sphere $\mathbb{CP}_1 \setminus \mathbf{e}$ gives rise to the action of the modular group $\mathrm{Mod}(\mathbf{e}) := QC(\mathbb{H}, \mathbf{e})/QC^0(\mathbb{H}, \mathbf{e})$ on the group $\pi_1(\mathbb{CP}_1 \setminus \mathbf{e}, \infty)/\ker \chi_\Lambda \cong \mathfrak{G}$. In fact, the action of f on the fundamental group depends only on the homotopy class of f, and the characterization of $\ker \chi_\Lambda$ used in the proof of Lemma 3 demonstrates its stability with respect to this action. For instance, for a smooth representative f of the homotopy class we have $f \cdot \ker \chi_\Lambda = \ker \chi_{f\Lambda} = \ker \chi_\Lambda$. The next result shows that the representation from the modular group into the automorphism group of $\mathfrak{G}$ is faithful.

Theorem 1. *The action of $f \in QC(\mathbb{H}, \mathbf{e})$ on the group $\mathfrak{G}$ is trivial if and only if $f \in QC^0(\mathbb{H}, \mathbf{e})$.*

Proof. Assume that f acts trivially on the group $\mathfrak{G}$. We shall show that the action of f on the fundamental group $\pi_1(\mathbb{CP}_1 \setminus \mathbf{e}, \infty)$ is also trivial. The fundamental group of the punctured sphere is generated by three classes of loops. These are loops in the punctured upper and lower half-planes and also loops in a sufficiently narrow punctured neighbourhood of the real axis. The action of f on the last class is trivial, and its actions on loops in the first two classes agree in view of the mirror symmetry $f(\overline{x}) = \overline{f}(x)$. We shall therefore analyse the action of f on the fundamental group $\pi_1(\mathbb{H} \setminus \mathbf{e}, \infty)$.

Each generator $[\lambda_s]$ of the fundamental group of the punctured half-plane produced by a labyrinth contains a loop λ_s without self-intersections such that its image $f\lambda_s$ is also a simple loop. The representation χ_Λ takes the generator $[\lambda_s]$ to an element G_s, $s \in \mathfrak{i}$. Recall that the group $\mathfrak{G}$ is freely generated by rank 2 groups, therefore it follows by Lemma 4 that $\chi_\Lambda[f\lambda_s] = G_s$ only in two cases: $[f\lambda_s] = [\lambda_s]$ and $[f\lambda_s] = [\lambda_s]^{-1}$. The second case cannot occur because f respects the orientation.

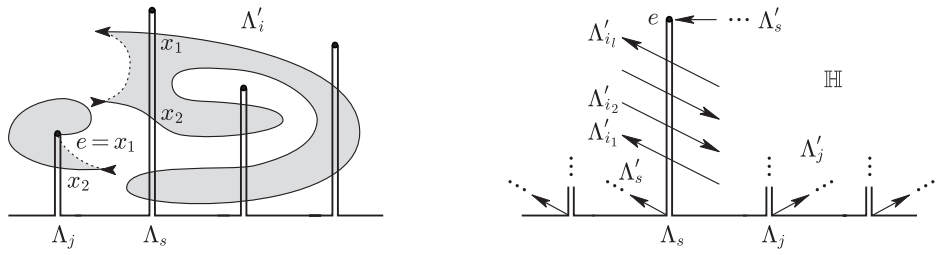
Having established that the action of f on the fundamental group of the punctured Riemann sphere is trivial we use a construction due to Ahlfors [5], [8]. Let $\mathbb{H} \to \mathbb{CP}_1 \setminus \mathbf{e}$ be the universal cover. A lift $\widetilde{f}\colon \mathbb{H} \to \mathbb{H}$ of f onto the covering space starting from an arbitrary point in the inverse image of the point at infinity commutes with covering transformations because the action of f on the fundamental group of the base is identical. Let $\widetilde{f}_t(u)$ be a point partitioning in the ratio $t : (1-t)$, $t \in [0,1]$, the non-Euclidean interval $[\widetilde{f}(u), u]$ in the Lobachevskiĭ plane. Lowering the map $\widetilde{f}_t(u)$ to the base we obtain a homotopy of $\mathbb{CP}_1 \setminus \mathbf{e}$ stabilizing infinity and connecting f with the identity map.

2.2.4. Equivalence of labyrinths.

Theorem 2. *Two labyrinths* $(\mathbf{e}, \Lambda)$ *and* $(\mathbf{e}, \Lambda')$ *are equivalent if and only if the induced representations* χ_Λ, $\chi_{\Lambda'}\colon \pi_1(\mathbb{CP}_1 \setminus \mathbf{e}, \infty) \to \mathfrak{G}$ *are the same.*

Proof. During a continuous deformation of the labyrinth Λ the representations χ_Λ into the discrete group $\mathfrak{G}$ must remain the same, therefore this representation is the same on equivalent labyrinths. Conversely, for $\chi_\Lambda = \chi_{\Lambda'}$ we shall explicitly describe the deformation $\Lambda' \to \Lambda$. In view of the mirror symmetry, such a deformation is uniquely defined by the motion of the labyrinth in the upper half-plane.

We start with the following preliminary observation: *the systems of free generators* (= *the alphabets*) $[\lambda'_i]$, $i \in \mathfrak{i}(\Lambda')$, $[\lambda_s]$, $s \in \mathfrak{i}(\Lambda)$, *of the fundamental group* $\pi_1(\mathbb{H} \setminus \mathbf{e}, \infty)$ *related to the labyrinths* Λ' *and* Λ *are the same.* For consider a simple loop representing a class $[\lambda'_i]$. Its irreducible factorization in the generators of the second system $[\lambda'_i] = [\lambda_{s_1}]^{\varepsilon_1}[\lambda_{s_2}]^{\varepsilon_2}\cdots[\lambda_{s_l}]^{\varepsilon_l}$ contains no repeating letters by Lemma 4. Accordingly, the word $G_{s_1}G_{s_2}\cdots G_{s_l} =: \chi_\Lambda([\lambda'_i]) = \chi_{\Lambda'}([\lambda'_i]) := G_i$ is irreducible. Such an equality in the group $\mathfrak{G}$ is possible if $[\lambda'_i] = [\lambda_i]^{\pm 1}$. The classes $[\lambda'_i]$ and $[\lambda_i]$ are conjugate in the fundamental group of the punctured hyperplane because the corresponding loops go counterclockwise about the same puncture. The elements $[\lambda_i]$ and $[\lambda_i]^{-1}$ cannot be conjugate in the freely generated groups, therefore $[\lambda'_i] = [\lambda_i]$. In particular, both labyrinths are of the same type: $\mathfrak{i}(\Lambda) = \mathfrak{i}(\Lambda')$. After the obvious deformation of Λ' we can assume that



(a) Deformation of the labyrinth Λ' (b) Expansion of the loop $\rho := \partial(\mathbb{H} \setminus \Lambda_s)$ in generators

Figure 4

the labyrinths are equal on the real axis, while in the upper half-plane they intersect transversally at finitely many points.

Assume that $\Lambda_s \cap \Lambda'_i$ contains points x_1 and x_2 in the upper half-plane with opposite orientation of the intersections and the segment of the arc Λ_s between them is disjoint from the labyrinth Λ'. The segments Λ_s and Λ'_i cut by the points x_1 and x_2 bound a cell in $\mathbb{H}$ disjoint from the labyrinth Λ' and, in particular, containing no punctures. This cell can be retracted — we depict the corresponding deformation in the background of Fig 4(a). We consider also the limiting case when one of the points x_1, x_2 is an end-point of Λ_s. Each of these deformations of Λ' reduces the number of its intersections with the labyrinth Λ. Hence in finitely many steps we arrive at an equivalent labyrinth of the above-described structure (still denoted by Λ') disjoint from Λ. We claim that the intersection of the two labyrinths in the upper half-plane now contains only points in $\mathbf{e}$. This actually means that Λ_i and Λ'_i, $i \in \mathrm{i}$, bound in $\mathbb{H}$ a cell containing no points from either labyrinth. Retracting such cells we obtain a deformation $\Lambda' \to \Lambda$.

Assume now that $\Lambda_s \setminus \mathbf{e}$ intersects the arcs $\Lambda'_{i_1}, \Lambda'_{i_2}, \dots, \Lambda'_{i_l}$ in the upper half-plane one after another, as in Fig. 4(b). We factor the loop ρ going along the boundary of $\mathbb{H} \setminus \Lambda_s$ in the two systems of representatives related to the labyrinths Λ and Λ'. Setting equal these expressions and taking account of the equality of the alphabets $[\lambda'_i] = [\lambda_i]$, $i \in \mathrm{i}$, we obtain a commutation relation in the freely generated group $\pi_1(\mathbb{H} \setminus \mathbf{e})$:

$$[\lambda_s] \cdot [\lambda_{i_1}]^{\varepsilon_1} [\lambda_{i_2}]^{\varepsilon_1} \cdots [\lambda_{i_l}]^{\varepsilon_l} = [\lambda_{i_1}]^{\varepsilon_l} [\lambda_{i_2}]^{\varepsilon_1} \cdots [\lambda_{i_l}]^{\varepsilon_1} \cdot [\lambda_s], \tag{10}$$

where $\varepsilon_j = \pm 1$ depending on the orientation of the intersection of Λ_s and Λ'_{i_j}. The word on the right-hand side of (10) is irreducible, for otherwise we would be able to make a deformation of the labyrinth Λ' described in the previous paragraph. Hence this word consists of one letter $[\lambda_s]$ and the labyrinth Λ' does not intersect the arc $\Lambda_s \cap \mathbb{H}$ at its interior points.

2.2.5. Quasiconformal deformation. The idea of the quasiconformal deformation of a group is due to Ahlfors and Bers [8], [9]; we merely adapt it to our aims. To start with, we fix an element $\{G^0_s\}^g_{s=0}$ of the deformation space $\mathcal{G}^k_g(\mathrm{i})$ generating the Kleinian group $\mathfrak{G}^0$ and a projection $x^0(u)\colon \mathcal{D}(\mathfrak{G}^0) \to \mathbb{CP}_1$ compatible with this group.

Construction. Let $f(x)$ be a quasiconformal motion of the plane of the complex variable x. We lift its Beltrami differential

$$\mu(x)\frac{\overline{dx}}{dx} := \frac{f_{\overline{x}}\overline{dx}}{f_x dx}$$

to the domain of discontinuity $\mathcal{D}(\mathfrak{G}^0)$ by means of the covering map compatible with the group. We extend the new differential

$$\widetilde{\mu}(u)\frac{\overline{du}}{du} := \mu(x^0(u))\frac{\overline{dx^0(u)}}{dx^0(u)}$$

by zero to the limit set of the group $\mathfrak{G}^0$, which has measure zero. Consider a quasiconformal homeomorphism $\widetilde{f}(u)$ of the Riemann sphere satisfying the Beltrami equation

$$\frac{\partial}{\partial \overline{u}}\widetilde{f}(u) = \widetilde{\mu}(u)\cdot\frac{\partial}{\partial u}\widetilde{f}(u) \tag{11}$$

and fixing three points: two fixed points of the generator G_0 and the distinguished point u_∞. The Beltrami differential $\widetilde{\mu}(u)\dfrac{\overline{du}}{du}$ is $\mathfrak{G}^0$-invariant by construction, therefore the uniqueness theorem for quasiconformal homeomorphisms states that the homeomorphisms $\widetilde{f}(u)$ and $\widetilde{f}(Gu)$ differ by a conformal motion of the Riemann sphere:

$$G^f \circ \widetilde{f} = \widetilde{f}\circ G, \qquad G\in\mathfrak{G}^0, \quad G^f\in \mathrm{PSL}_2(\mathbb{C}). \tag{12}$$

The group $\mathfrak{G}^f := \widetilde{f}\mathfrak{G}^0\widetilde{f}^{-1}$ generated by such motions is isomorphic to $\mathfrak{G}^0$, acts discontinuously in the domain $\mathcal{D}(\mathfrak{G}^f) := \widetilde{f}\mathcal{D}(\mathfrak{G}^0)$, and is called the *quasiconformal deformation* of the group $\mathfrak{G}^0$ generated by the element $f\in QC(\mathbb{H})$. The distinguished system of generators $\{G^0_s\}^g_{s=0}$ of $\mathfrak{G}^0$ is taken to a distinguished system of generators $\{G^f_s := \widetilde{f}G^0_s\widetilde{f}^{-1}\}^g_{s=0}$.

We now formulate two technical results underlining the natural character of the construction of the quasiconformal deformation. They are an easy consequence of the uniqueness of the normalized quasiconformal map with fixed Beltrami coefficient [8].

Lemma 5. (1) *If $f\in QC(\mathbb{H})$ deforms the group $\mathfrak{G}^0$ into $\mathfrak{G}^f$, then the distinguished holomorphic projection $x^f(u)\colon \mathcal{D}(\mathfrak{G}^f)\to\mathbb{CP}_1$ with normalization $x^f(u) := f\circ x^0(u)$, $u\in\{\operatorname{fix} G_0, u_\infty\}$, completes the diagram* (a) *to a commutative diagram*:

$$\text{(a)}\quad \begin{array}{ccc} \mathcal{D}(\mathfrak{G}^0) & \xrightarrow{\widetilde{f}} & \mathcal{D}(\mathfrak{G}^f) \\ \downarrow{\scriptstyle x^0} & & \downarrow{\scriptstyle x^f} \\ \mathbb{CP}_1 & \xrightarrow{f} & \mathbb{CP}_1 \end{array}, \qquad \text{(b)}\quad \begin{array}{ccccc} \mathcal{D}(\mathfrak{G}^0) & \xrightarrow{\widetilde{h}} & \mathcal{D}(\mathfrak{G}^h) & \xrightarrow{\widetilde{f}} & \mathcal{D}(\mathfrak{G}^{fh}) \\ \downarrow{\scriptstyle x^0} & & \downarrow{\scriptstyle x^h} & & \downarrow{\scriptstyle x^{fh}} \\ \mathbb{CP}_1 & \xrightarrow{h} & \mathbb{CP}_1 & \xrightarrow{f} & \mathbb{CP}_1 \end{array}. \tag{13}$$

(2) *The deformation of the group $\mathfrak{G}^0$ by the composite fh of two maps can be performed in two steps: first, by means of h one deforms the group $\mathfrak{G}^0$ into $\mathfrak{G}^h$ (the left square in* (b)*) and uses it as the distinguished group for the second deformation, by means of f (the right square in* (b)*). The deforming homeomorphism of the composite map fh is equal to $\widetilde{f}\widetilde{h}$ and the required deformation of the group is $\mathfrak{G}^{fh} = \widetilde{f}\mathfrak{G}^h\widetilde{f}^{-1}$.*

Remark. The second result can be understood as follows. Quasiconformal maps act on the left on pairs consisting of a system of generators of a Kleinian group and a holomorphic projection onto the sphere: $f\cdot(\{G_s^0\}_{s=0}^g, x^0) := (\{G_s^f\}_{s=0}^g, x^f)$. This action is associative: $(fh)\cdot(\{G_s^0\}_{s=0}^g, x^0) = f\cdot(\{G_s^h\}_{s=0}^g, x^h) =: f\cdot(h\cdot(\{G_s^0\}_{s=0}^g, x^0))$.

2.3. Equivalence of the representations. The equivalence of the four definitions of the universal cover of the moduli space means the existence of (compatible) continuous bijections between the topological spaces $\widetilde{\mathcal{H}}_g^k$, $\mathcal{T}_g^k$, $\mathcal{G}_g^k$, $\mathcal{L}_g^k$ introduced above. In the following three subsections we shall successively establish homeomorphisms: between the Teichmüller space and the deformation space of the Kleinian group, between the Teichmüller space and the universal cover of the moduli space, between the labyrinth space and the deformation space of the group. We now describe these correspondences at an intuitive level.

$\mathcal{T}_g^k \leftrightarrow \mathcal{L}_g^k$. A smooth representative f of a Teichmüller class takes the distinguished labyrinth $(\mathbf{e}^0, \Lambda^0)$ to a labyrinth $(f\mathbf{e}^0, f\Lambda^0)$ of the same type. Two labyrinths of the same type can always be transformed into one another by a suitable map f, which is unique up to an isotopy of the punctured plane.

$\widetilde{\mathcal{H}}_g^k(\mathbf{e}^0) \leftrightarrow \mathcal{L}_g^k$. The universal cover of the moduli space is the set of the branching divisors $\mathbf{e}$ together with the history of their motion from the distinguished divisor $\mathbf{e}^0$. One can recover this history by treating cuts in the labyrinth as the traces left by points in the divisor in their motion. Conversely, each path in the moduli space can be deformed so that the points in the divisor $\mathbf{e}$ moving in the complex plane do not intersect their traces. The resulting picture in the plane can be completed to a labyrinth.

2.3.1. The isomorphy of $\mathcal{T}_g^k$ and $\mathcal{G}_g^k$. We fix an element $\{G_s^0\}_{s=0}^g$ in the deformation space $\mathcal{G}_g^k(\mathfrak{i})$ and a compatible cover $x^0(u)$. The following result is typical for the theory of Teichmüller spaces [9], [5].

Theorem 3. *Quasiconformal deformations bring about a homeomorphism between $\mathcal{G}_g^k(\mathfrak{i})$ and the Teichmüller space $\mathcal{T}_g^k(\mathbf{e}^0)$, where $\mathbf{e}^0 := \{x^0(\operatorname{fix} G_s^0)\}_{s=0}^g$ are the branching points of the cover $x^0(u)$.*

The proof of Theorem 3 splits naturally into several steps.

Lemma 6. *For $f \in QC(\mathbb{H})$ the quasiconformal deformation $\{G_s^f\}_{s=0}^g$ of the distinguished element lies in the same space $\mathcal{G}_g^k(\mathfrak{i})$. The cover $x^f(u)$ generated by f is compatible with the deformed group $\mathfrak{G}^f$.*

Proof. The following objects are mirror-symmetric relative to the real axis: the Beltrami coefficient $\mu(x) = \overline{\mu(\overline{x})}$ of the function f, the regular cover $x^0(u) = \overline{x^0(\overline{u})}$, and the normalization set: $(\pm i, \infty)$ if $0 \in \mathfrak{i}$ or $(0, 1, \infty)$ if $0 \in \mathfrak{i}'$. The uniqueness theorem for the normalized quasiconformal homeomorphism ensures that $\widetilde{f}(u) \in QC(\mathbb{H})$. Hence the new generators $G_s^f := \widetilde{f} G_s^0 \widetilde{f}^{-1}$, $s = 0, 1, \dots, g$, are real. The deformation of G_0 is always trivial because $\widetilde{f}$ stabilizes its fixed points. We claim that the deformations G_s^f of the other rotations satisfy the geometric condition in § 2.1.2. The motion $\widetilde{f}$ fixes the end-points of the interval $(0, u_\infty)$. In fact, if $0 \in \mathfrak{i}$, then $\widetilde{f}(0) = \widetilde{f} G_0(\infty) = G_0^f \widetilde{f}(\infty) = 0$. Hence $\widetilde{f}$ takes the real diameters of the circles

$C_1, C_2, \dots, C_g$ into disjoint closed subintervals of $(0, u_\infty)$, and we construct new circles $C_1^f, C_2^f, \dots, C_g^f$ with these subintervals as diameters. The generator G_s^f, $s = 1, \dots, g$, maps the circle C_s^f into itself, but reverses the orientation, therefore G_s^f has two fixed points on C_s^f. They are real if $s \in \mathfrak{i}'$ and complex conjugate if $s \in \mathfrak{i}$. Hence the new rotations G_s^f, $s = 0, \dots, g$, satisfy the above-mentioned geometric condition and define an element of the same deformation space $\mathcal{G}_g^k(\mathfrak{i})$.

The deformation f gives rise to the branched cover $x^f(u)$ with pole at $u = u_\infty$ in accordance with the normalization. It is symmetric relative to the real axis and respects the orientation of the real axis in the neighbourhood of the pole $u = u_\infty$ because the other three maps in the commutative diagram (13)(a) have this property.

Lemma 7. *Two maps in $QC(\mathbb{H})$ deform the distinguished element in the same fashion if and only if they belong to the same class in the Teichmüller space $\mathcal{T}_g^k(\mathbf{e}^0)$.*

Proof. (1) We start with a special case of the lemma when one of the quasiconformal maps is identical.

1(a) *If $f \in \mathfrak{A}_1^+ \cdot QC^0(\mathbb{H}, \mathbf{e}^0)$, then f does not deform the generators of the group $\mathfrak{G}^0$.* The deformation of the distinguished group by $f \in QC(\mathbb{H}, \mathbf{e}^0)$ can in fact be explicitly found. Up to normalization it is equal to the action of the modular group and, in particular, it is trivial for $f \in QC^0(\mathbb{H}, \mathbf{e}^0)$. The left action of the affine group on f preserves the Beltrami coefficient of the map and therefore has no effect on the deformation $\widetilde{f}$.

Making punctures in the domain $\mathcal{D}$ of discontinuity of the distinguished group at the fixed points of the elliptic transformations we obtain a space $\overset{\circ}{\mathcal{D}}$ with a free action of the covering transformation group $\mathfrak{G}^0$. We lift the map $f \in QC(\mathbb{H}, \mathbf{e}^0)$ to an automorphism of the covering space $\overset{\circ}{\mathcal{D}}$ from the distinguished point u_∞:

$$\begin{array}{ccc} \overset{\circ}{\mathcal{D}} & \xrightarrow{\breve{f}} & \overset{\circ}{\mathcal{D}} \\ \downarrow{\scriptstyle x^0} & & \downarrow{\scriptstyle x^0} \\ \mathbb{CP}_1 \setminus \mathbf{e}^0 & \xrightarrow{f} & \mathbb{CP}_1 \setminus \mathbf{e}^0 \end{array} . \tag{14}$$

The resulting map $\breve{f}$ can be defined by continuity at the punctures in the domain of discontinuity and on the limit set of the distinguished group with the help of the equivariance condition $\breve{f}G = (f \cdot G)\breve{f}$, $G, (f \cdot G) \in \mathfrak{G}^0$. The action of $G \to f \cdot G$ on the covering transformation group $\mathfrak{G}^0$ is induced by the action of f on the fundamental group of the punctured plane $\mathbb{CP}_1 \setminus \mathbf{e}^0$; see § 2.2.3. The homeomorphism $\breve{f}\colon \mathbb{CP}_1 \to \mathbb{CP}_1$ is quasiconformal and its Beltrami coefficient is $\widetilde{\mu}$. Hence $\breve{f}$ differs from $\widetilde{f}$ in (13)(a) only by normalization. Calculating $\breve{f}$ at the fixed points of G_0 we obtain

$$\widetilde{f} = \alpha\breve{f}, \quad G_s^f := \widetilde{f}G_s^0\widetilde{f}^{-1} = \alpha(f \cdot G_s^0)\alpha^{-1}, \qquad \alpha \in \mathfrak{A}_1^+, \tag{15}$$

where the affine motion $\alpha = \mathrm{id}$ if $0 \in \mathfrak{i}'$, while for $0 \in \mathfrak{i}$ it is defined by the condition $\alpha\breve{f}(\pm i) := \pm i$.

1(b) *If f does not deform the generators of $\mathfrak{G}^0$, then $f \in \mathfrak{A}_1^+ \cdot QC^0(\mathbb{H}, \mathbf{e}^0)$.* Let $\mathfrak{G}^f = \mathfrak{G}^0$ in the deformation diagram (13)(a); then $x^f = \alpha x^0$ for some $\alpha \in \mathfrak{A}_1^+$. Multiplying f by an affine motion on the left we can assume that $f \in QC(\mathbb{H}, \mathbf{e}^0)$ and $x^f = x^0$. Then the deformation of the generators of the distinguished group is defined by formula (15) for $\alpha = \mathrm{id}$. By Theorem 1 it follows from the equalities $f \cdot G_s^0 = G_s^0$ that $f \in QC^0(\mathbb{H}, \mathbf{e}^0)$.

(2) We reduce the general case of the lemma to the special case of part (1). Combining the deformation diagrams of the maps $f_1, f_2 \in QC(\mathbb{H})$ we obtain

$$\begin{array}{ccccc} \mathcal{D}(\mathfrak{G}^{f_1}) & \xleftarrow{\widetilde{f}_1} & \mathcal{D}(\mathfrak{G}^0) & \xrightarrow{\widetilde{f}_2} & \mathcal{D}(\mathfrak{G}^{f_2}) \\ \downarrow{\scriptstyle x^{f_1}} & & \downarrow{\scriptstyle x^0} & & \downarrow{\scriptstyle x^{f_2}} \\ \mathbb{CP}_1 & \xleftarrow{f_1} & \mathbb{CP}_1 & \xrightarrow{f_2} & \mathbb{CP}_1 \end{array} . \tag{16}$$

By Lemma 5 on composite deformations the deformations of the generators of the distinguished group $\mathfrak{G}^0$ are equal if and only if the map $f_2 f_1^{-1}$ does not deform the generators of $\mathfrak{G}^{f_1}$. By the special case considered above this can occur if and only if $f_2 f_1^{-1} \in \mathfrak{A}_1^+ \cdot QC^0(\mathbb{H}, f_1\mathbf{e}^0)$, which is equivalent to the result of the lemma in the general case: $f_2 \in \mathfrak{A}_1^+ \cdot f_1 \cdot QC^0(\mathbb{H}, \mathbf{e}^0)$.

Proof of Theorem 3. We showed in Lemmas 6 and 7 that the map $\mathcal{T}_g^k(\mathbf{e}^0) \to \mathcal{G}_g^k(\mathfrak{i})$ is well defined and injective; we now discuss its surjectivity. One can find a quasiconformal map $\widetilde{f}(u)$ of the standard fundamental domain $\mathcal{R}$ of the distinguished element of $\mathcal{G}_g^k(\mathfrak{i})$ onto the fundamental domain of an arbitrary element of the same space that

(a) commutes with the reflection $\overline{J}u := \overline{u}$;
(b) respects the identifications at the boundary and
(c) fixes the three points $\operatorname{fix} G_0$, u_∞.

One can construct such a map explicitly, but the corresponding description is lengthy. Extending $\widetilde{f}$ by means of equivariance condition (12) to the whole of the Riemann sphere one obtains the map f in the bases in the diagram (13)(a). By the uniqueness of the normalized quasiconformal map f deforms the distinguished element into the prescribed one.

The continuity of the bijection $\mathcal{T}_g^k \leftrightarrow \mathcal{G}_g^k$ with respect to the Teichmüller metric in $\mathcal{T}_g^k$ and the topology of the cell $\mathcal{G}_g^k$ follows by the construction of the direct and the inverse maps; see also formula (17) for infinitesimal quasiconformal transformations.

2.3.2. Isomorphy between $\mathcal{T}_g^k$ and $\widetilde{\mathcal{H}}_g^k$. We shall identify the orbit manifold of the modular group with the moduli space.

Lemma 8. *The spaces $\mathcal{T}_g^k(\mathbf{e}^0)/\operatorname{Mod}(\mathbf{e}^0)$ and $\mathcal{H}_g^k$ are homeomorphic.*

Proof. We assign to the map $f \in QC(\mathbb{H})$ the symmetric divisor $f\mathbf{e}^0$. This defines an injection of the orbit space of the modular group acting on the Teichmuller space into the moduli space. We shall show that this map is

(1) continuous in the natural topology and
(2) has a continuous inverse in the entire moduli space.

This will prove the homomorphy of the two spaces.

(1) For local coordinate variables in the moduli space in the neighbourhood of the distinguished point $\mathbf{e}^0$ one can take the independent real and imaginary parts of the variable points in the divisor $\{e_1, \dots, e_{2g}, -1, 1\} =: \mathbf{e} \approx \mathbf{e}^0$ of the type (g, k): we have fixed the pair of points ± 1 in the divisor assuming that $k > 0$. In the orbit space of the modular group $\mathfrak{A}_1^+ \setminus QC(\mathbb{H})/QC(\mathbb{H}, \mathbf{e}^0)$ one defines the Teichmuller metric

$$\rho([f], [h]) := \inf_{[f],[h]} \log \frac{1 + \|\mu(x)\|_\infty}{1 - \|\mu(x)\|_\infty},$$

where the infimum is taken over the representatives $f_1 \in [f], h_1 \in [h]$ of the classes, $\mu(x)$ is the Beltrami coefficient of the map $f_1 h_1^{-1}$, the norm $\|\mu(x)\|_\infty$ is the essential maximum of $\mu(x)$ in the plane. We can verify that the embedding $\mathcal{T}_g^k(\mathbf{e}^0)/\operatorname{Mod}(\mathbf{e}^0) \to \mathcal{H}_g^k$ is continuous in the neighbourhood of [id]; the general case follows by replacing the distinguished divisor $\mathbf{e}^0$. There exists a formula for the 'principal part' of the quasiconformal map $f(x)$ with small Beltrami coefficient $\mu(x)$ and fixed points $\pm 1, \infty$ [5], [6], [8], [9]:

$$f(e) - e = \frac{1}{2\pi i} \int_{\mathbb{C}} \mu(x) \frac{e^2 - 1}{x^2 - 1} \frac{dx \wedge d\overline{x}}{x - e} + O(\|\mu\|_\infty^2) \tag{17}$$

where the remainder term has a uniform estimate on compact subsets of the plane. It is clear from (17) that the classes $[f]$ close to [id] in the Teichmüller metric only slightly deform the divisor $\mathbf{e}^0$.

(2) Let $\sigma(x)$ be an infinitely smooth cut-off function with support small by comparison with the distance between the points in $\mathbf{e}^0$, equal to 1 in a (complex) neighbourhood of $x = 0$, and conjugation invariant: $\sigma(\overline{x}) = \sigma(x)$. In a small neighbourhood of the distinguished point $\mathbf{e}^0$ in the moduli space we define a local section of the projection $QC(\mathbb{H}) \to \mathcal{H}_g^k$:

$$f(\mathbf{e}, x) := x + \sum_{s=1}^{2g} (e_s - e_s^0) \sigma(x - e_s^0), \qquad x \in \mathbb{C}. \tag{18}$$

The map (18) is a local inversion of our embedding $\mathcal{T}_g^k(\mathbf{e}^0)/\operatorname{Mod}(\mathbf{e}^0) \to \mathcal{H}_g^k$ and it is continuous. Similar sections can be constructed in the neighbourhood of each point in the moduli space. Taking composites one defines a continuous inverse map $\mathcal{H}_g^k \to \mathcal{T}_g^k(\mathbf{e}^0)/\operatorname{Mod}(\mathbf{e}^0)$ in the neighbourhood of an arbitrary prescribed point.

Theorem 4. (i) *The spaces* $\mathcal{T}_g^k(\mathbf{e}^0)$ *and* $\widetilde{\mathcal{H}}_g^k(\mathbf{e}^0)$ *are homeomorphic.*

(ii) *the groups* $\operatorname{Mod}(\mathbf{e}^0)$ *and* $\operatorname{Br}_{g-k+1}$ *are isomorphic.*

Proof. We have just shown that the moduli space $\mathcal{H}_g^k$ is the quotient of $\mathcal{T}_g^k(\mathbf{e}^0)$ by the modular group action. We claim that $\operatorname{Mod}(\mathbf{e}^0)$ acts (1) freely and (2) discontinuously in the Teichmüller space and therefore the projection $\mathcal{T}_g^k(\mathbf{e}^0) \to \mathcal{H}_g^k$ is a cover. This is a universal cover because the space $\mathcal{T}_g^k(\,\cdot\,) \cong \mathcal{G}_g^k(\,\cdot\,)$ is a cell by Lemma 2. Accordingly, the covering transformation group $\operatorname{Mod}(\mathbf{e}^0)$ of the cover is isomorphic to the fundamental group of the moduli space, which by Lemma 1 is isomorphic to the braid group $\operatorname{Br}_{g-k+1}$.

(1) For a representative $h \in QC(\mathbb{H}, \mathbf{e}^0)$ of the modular group assume that there exists a motion $f \in QC(\mathbb{H})$ such that $fh \in \alpha f QC^0(\mathbb{H}, \mathbf{e}^0)$ for some $\alpha \in \mathfrak{A}_1^+$.

Then $\alpha f\mathbf{e}^0 = f\mathbf{e}^0$ and since $\mathfrak{A}_1^+$ acts freely on the divisor set, it follows that $\alpha = \mathrm{id}$ and therefore $h \in QC^0(\mathbb{H}, \mathbf{e}^0)$. Thus, h represents the identity element of the modular group.

(2) Transformations in the modular group are isometries of the Teichmüller space, therefore the discontinuity of the action of $\mathrm{Mod}(\mathbf{e}^0)$ follows from the fact that it has discrete orbits. We recall from § 2.1.1 that each orbit of the modular group can, by replacing the distinguished divisor $\mathbf{e}^0$, be isometrically transformed into an orbit passing through the distinguished point [id] in the Teichmüller space. Thus, let $h_n \in QC(\mathbb{H}, \mathbf{e}^0)$ be a sequence with infinitesimally small deformations of the generators of the distinguished group: $G_s^{h_n} \to G_s^0$, $s = 0, \dots, g$.

(2a) If $0 \in \mathfrak{i}'$, then the deformation of generators calculated in (15) is as follows: $G_s^{h_n} = h_n \cdot G_s^0$. In the discontinuity domain of the distinguished group we select a point u with trivial isotropy group (for instance, $u = u_\infty$). The convergence $G_s^{h_n} u \to G_s^0 u$ shows that $h_n \cdot G_s^0 = G_s^0$ starting from some n. By Theorem 1 the trivial action of h_n on the group means that h_n represents the unit element of the modular group.

(2b) If $0 \in \mathfrak{i}$, then the deformation of the generators is $G_s^{h_n} = \alpha_n(h_n \cdot G_s^0)\alpha_n^{-1}$, $\alpha_n \in \mathfrak{A}_1^+$. The distinguished group contains an element invariant under all the h_n; it corresponds to the class of the loop $[\mathbb{R}+i0]$ in the fundamental group $\pi_1(\mathbb{CP}_1 \setminus \mathbf{e}^0, \infty)$ and is equal to $G^* = \prod_{s\in\mathfrak{i}} G_s^0$, where the product is ordered in increasing order of the indices. One can deduce from the convergence $\alpha_n G^* \alpha_n^{-1} \to G^*$ that $\alpha_n \to \mathrm{id}$. The rest of the proof proceeds as in part (2a) of the proof.

The identification of the Teichmüller space with the universal cover of the moduli space and the deformation space of the special Kleinian group gives rise to two local coordinate charts in $\mathcal{T}_g^k$. First, we have the global coordinate variables $(c_s, r_s)_{s=1}^g$ in $\mathcal{G}_g^k$ ranging over a cell. Second, we have systems of coordinates in $\mathcal{H}_g^k$ related to the branch points. The following result establishes a connection between the two systems of coordinates.

Theorem 5. *The map $\widetilde{\mathcal{H}}_g^k(\mathbf{e}^0) \to \mathcal{G}_g^k(\mathfrak{i})$ is real analytic in local coordinates. Its Jacobi matrix is non-degenerate with entries effectively calculated with the use of quadratic Poincaré series.*

Proof. To express our map in terms of local variables one must deform the generators of the distinguished group by means of the local section (18) of the projection in Lemma 8. It is sufficient to study the map in the neighbourhood of the distinguished divisor $\mathbf{e}^0$ and, if necessary, to replace the distinguished divisor together with the distinguished group with the help of Lemma 5 (§ 2) on composite deformations. For a change let $0 \in \mathfrak{i}$, that is, assume that the distinguished divisor $\mathbf{e}^0$ contains complex conjugate points, for instance, $\pm i$. For local coordinate variables in the neighbourhood of the distinguished point in the moduli space $\mathcal{H}_g^k$ we shall take the (independent) real and imaginary parts of the complex points $e_1, e_2, \dots, e_{2g}$, which together with $\{\pm i\}$ form a simple branching divisor $\mathbf{e} \approx \mathbf{e}^0$. The fixed points of the generators of the distinguished group deformed by $f(\mathbf{e}, x)$ define a map $\{e_s\}_{s=1}^{2g} \to \{c_s, r_s\}_{s=1}^g$ in a small *complex* neighbourhood of $\{e_s^0\}_{s=1}^{2g}$:

$$\left.\begin{aligned} c_s(\mathbf{e}) \pm r_s(\mathbf{e}), &\quad s \in \mathfrak{i}', \\ c_s(\mathbf{e}) \pm i r_s(\mathbf{e}), &\quad s \in \mathfrak{i}, \end{aligned}\right\} := \operatorname{fix} G_s^f = \widetilde{f}(\mathbf{e}, \operatorname{fix} G_s^0), \quad s = 1, \dots, g. \tag{19}$$

Corresponding to symmetric divisors $\mathbf{e}$ in the $2g$-dimensional complex neighbourhood there are a real $2g$-plane and the real moduli c_s, $r_s > 0$. We claim that (1) the complex map $\{e_s\}_{s=1}^{2g} \to \{c_s, r_s\}_{s=1}^{g}$ is holomorphic, (2) its Jacobi matrix can be explicitly calculated and (3) it is non-singular.

(1) The Beltrami coefficient $\mu(\mathbf{e}, x)$ of the function (18) depends holomorphically in $L_\infty(\mathbb{C})$ on the components of $\mathbf{e}$. The dependence on $\mathbf{e}$ of the coefficient

$$\widetilde{\mu}(\mathbf{e}, u) := \mu(\mathbf{e}, x^0(u))\frac{\overline{dx^0(u)}}{dx^0(u)}$$

is also holomorphic; the latter is now defined in the domain of discontinuity $\mathcal{D}(\mathfrak{G}^0)$. By a well-known result [8] the map $\widetilde{f}(\mathbf{e}, \cdot\,)$ depends analytically on the parameters $\mathbf{e}$. In particular, all functions $v(\mathbf{e}) := \widetilde{f}(\mathbf{e}, v)$, $v \in \{\operatorname{fix} G_s^0\}_{s=1}^{g}$ are holomorphic; the functions $c_s(\mathbf{e})$ and $r_s(\mathbf{e})$ are linear combinations of them.

(2) The differentials of the functions $v(\mathbf{e})$ can be calculated by the formula for an infinitesimal deformation [8], [5], in which $\overset{o}{=}$ means equality to within terms of order $O\bigl(\sum_{s=1}^{2g} |e_s - e_s^0|^2\bigr)$:

$$\begin{aligned}
2\pi i(v(\mathbf{e}) - v(\mathbf{e}^0)) &\overset{o}{=} \int_{\mathbb{C}} \widetilde{\mu}(\mathbf{e}, u)\frac{v^2+1}{u^2+1}\,\frac{du \wedge \overline{du}}{u - v} \\
&= \int_{\mathcal{R}} \widetilde{\mu}(\mathbf{e}, u)\biggl[\sum_{G \in \mathfrak{G}^0} \frac{(dG(u)/du)^2}{(Gu)^2+1}\,\frac{v^2+1}{Gu - v}\biggr] du \wedge \overline{du} \\
&= (v^2+1)\int_{\mathbb{C}} \mu(\mathbf{e}, x)\frac{P_{2g-1}^v(x)}{w^2(x)}\, dx \wedge \overline{dx}. \qquad (20)
\end{aligned}$$

In the next to last integral the quadratic Poincaré series defines on the Riemann sphere $x^0(\mathcal{R})$ a meromorphic quadratic differential $w^{-2}P_{2g-1}^v(x)\,(dx)^2$ dependent on the point $v \in \{\operatorname{fix} G_s^0\}_{s=1}^{g}$ as a parameter. Its singularities are simple poles, which can be placed at points in the divisor $\mathbf{e}^0$ and at infinity. Such quadratic differentials (of finite area [5]) form a complex vector space of dimension $2g$ with basis

$$\Omega_s(x)(dx)^2 := \frac{(dx)^2}{(x^2+1)(x - e_s^0)}\,, \qquad s = 1, \dots, 2g. \qquad (21)$$

We expand our quadratic differential $w^{-2}P_{2g-1}^v(x)\,(dx)^2$ with respect to the basis (21) with coefficients $a_s^v := P_{2g-1}^v(e_s^0)\prod_{j=1, j\neq s}^{2g}(e_s^0 - e_j^0)^{-1}$ and continue equality (20) as follows:

$$\begin{aligned}
&(v^2+1)\int_{\mathbb{C}} \mu(\mathbf{e}, x)\sum_{s=1}^{2g} a_s^v \Omega_s(x)\, dx \wedge \overline{dx} \\
&\quad \overset{o}{=} (v^2+1)\int_{\mathbb{C}} \sum_{s=1}^{2g} a_s^v \Omega_s(x) \sum_{j=1}^{2g}(e_j - e_j^0)\sigma_{\overline{x}}(x - e_j^0)\, dx \wedge \overline{dx} \\
&\quad = (v^2+1)\sum_{j=1}^{2g}(e_j - e_j^0)\int_{\operatorname{supp}\sigma_{\overline{x}}(x - e_j^0)} d\biggl(-\sum_{s=1}^{2g} a_s^v \Omega_s(x)\sigma(x - e_j^0)\, dx\biggr) \\
&\quad = 2\pi i\sum_{s=1}^{2g} a_s^v \frac{v^2+1}{(e_s^0)^2+1}(e_s - e_s^0).
\end{aligned}$$

One can recover the differentials of the functions $c_s(\mathbf{e})$ and $r_s(\mathbf{e})$ from these expressions.

(3) Assume that the differential of the map $\{e_s\}_{s=1}^{2g} \to \{c_s, r_s\}_{s=1}^{g}$ is singular at $\mathbf{e}^0$; then there exists a non-trivial tangent vector $\xi := \sum_{j=1}^{2g} \varepsilon_j \partial/\partial e_j$ such that for $\mathbf{e} = \mathbf{e}^0$,

$$\xi c_s(\mathbf{e}) = \xi r_s(\mathbf{e}) = 0, \qquad s = 1, 2, \dots, g.$$

We now differentiate the condition of the equivariance of $\widetilde{f}(\mathbf{e}, \cdot\,)$:

$$G^f \circ \widetilde{f}(\mathbf{e}, u) = \widetilde{f}(\mathbf{e}, u) \circ G, \qquad G \in \mathfrak{G}^0, \quad G^f \in \mathfrak{G}^f,$$

in the direction of ξ. We see that at the point $\mathbf{e}^0$ under consideration the velocity of the deformation $\widetilde{f}(\mathbf{e}, u)$ defines a $\mathfrak{G}^0$-invariant inverse differential $\xi \widetilde{f}(\mathbf{e}^0, u)(du)^{-1}$. We claim that all the coefficients ε_s of the vector ξ vanish. For a proof we lift the basis elements in (21) to $\mathcal{D}(\mathfrak{G}^0)$: $\Omega_s(x^0(u))(dx^0(u))^2 =: \widetilde{\Omega}_s(u)(du)^2$. The product of the quadratic and the inverse differential is a smooth $\mathfrak{G}^0$-invariant differential $\widetilde{\Omega}_s(u) \cdot \xi \widetilde{f}(\mathbf{e}^0, u)\, du$ on $\mathcal{D}(\mathfrak{G}^0)$, therefore

$$\begin{aligned}
0 &= \int_{\partial \mathcal{R}(\mathfrak{G}^0)} \widetilde{\Omega}_s(u) \cdot \xi \widetilde{f}(\mathbf{e}^0, u)\, du = \int_{\mathcal{R}(\mathfrak{G}^0)} d\big(\widetilde{\Omega}_s \cdot \xi \widetilde{f}\, du\big) \\
&= -\int_{\mathcal{R}(\mathfrak{G}^0)} \widetilde{\Omega}_s \cdot (\xi \widetilde{f})_{\overline{u}}\, du \wedge \overline{du} \\
&\quad \text{(differentiations of } \widetilde{f} \text{ with respect to } e_s \text{ and with respect to the variable } \overline{u} \\
&\quad \text{are interchangeable)} \\
&= -\int_{\mathcal{R}(\mathfrak{G}^0)} \widetilde{\Omega}_s(u) \sum_{j=1}^{2g} \varepsilon_j \sigma_{\overline{x}}(x(u) - e_j^0) \frac{\overline{\dot{x}(u)}}{\dot{x}(u)}\, du \wedge \overline{du} \\
&= -\int_{\mathbb{C}} \Omega_s(x) \sum_{j=1}^{2g} \varepsilon_j \sigma_{\overline{x}}(x - e_j^0)\, dx \wedge \overline{dx} \\
&= \sum_{j=1}^{2g} \varepsilon_j \int_{\operatorname{supp} \sigma_{\overline{x}}(x - e_j^0)} d\big(\Omega_s(x) \sigma(x - e_j^0)\, dx\big) \\
&= -2\pi i \sum_{j=1}^{2g} \varepsilon_j \operatorname*{Res}_{x = e_j^0} \Omega_s(x) = -2\pi i \frac{\varepsilon_s}{(e_s^0)^2 + 1}\,.
\end{aligned}$$

Since the quantity e_s^0 is finite, it follows that $\varepsilon_s = 0$.

2.3.3. Isomorphy of $\mathcal{L}_g^k$ and $\mathcal{G}_g^k$. In finding the group of the branched cover $x(u)$ in § 2.2.2 we already assigned to an element of the deformation space a special labyrinth, the projection of the boundary of the fundamental domain defined by this element. The inversion of this correspondence is the basis of the following result.

Theorem 6. *The spaces $\mathcal{L}_g^k(\mathrm{i})$ and $\mathcal{G}_g^k(\mathrm{i})$ are homeomorphic.*

Proof. We shall establish a 1-1 correspondence between these spaces.

(1) *The map* $\mathcal{G}_g^k(\mathfrak{i}) \to \mathcal{L}_g^k(\mathfrak{i})$. Corresponding to each element of the deformation space $\mathcal{G}_g^k(\mathfrak{i})$ is a system of circles $C_0, C_1, \dots, C_g$ bounding the fundamental domain $\mathcal{R}$ generated by this element of the Kleinian group. The pole u_∞ of the cover $x(u)$ compatible with the group lies in $\mathcal{R}$ if $0 \in \mathfrak{i}'$. For $0 \in \mathfrak{i}$ this can be attained by the replacement of the 'infinitely large' circle C_0 by a 'very large' circle $C_\varepsilon := \{u \in \mathbb{C} : |\varepsilon u + 1|^2 = \varepsilon^2 + 1\}$ with small $\varepsilon > 0$. The projection $x(u)$ takes the boundary of the fundamental domain to a labyrinth Λ of type $(g, k, \mathfrak{i})$, which does not change its class after admissible perturbations of the circles C_i, $i \in \mathfrak{i}$.

(2) *The map* $\mathcal{L}_g^k(\mathfrak{i}) \to \mathcal{G}_g^k(\mathfrak{i})$. For definiteness let $0 \in \mathfrak{i}'$; there are fewer technical details in this case. Corresponding to each divisor $\mathbf{e}$ of type (g, k) is an orbit of the modular group acting in $\mathcal{G}_g^k(\mathfrak{i})$. All points in this orbit are associated with the same compatible cover $x(u)\colon (\overset{\circ}{\mathcal{D}}, u_\infty) \to (\mathbb{CP}_1 \setminus \mathbf{e}, \infty)$. By Lemma 3 the group of this cover is equal to the kernel of each representation χ_Λ from the fundamental group $\mathbb{CP}_1 \setminus \mathbf{e}$ into the abstract group $\mathfrak{G} := \langle G_s,\ s \in \mathfrak{i} \cup \mathfrak{i}' : G_s^2 = 1\rangle$. Fixing a labyrinth $(\mathbf{e}, \Lambda)$ one can therefore realize elements of $\mathfrak{G}$ as covering transformations of $x(u)$, that is, linear fractional maps in $\mathrm{PGL}_2(\mathbb{R})$. For instance, $G_0(u) = -u$ (= the unique rotation of order 2 with fixed points $0, \infty$). We shall show that the realization of the remaining generators $G_1, G_2, \dots, G_g$ satisfies the geometric condition in § 2.1.2.

The fundamental group of the Riemann sphere cut along the labyrinth Λ lies in the kernel of χ_Λ by the definition of the representation. Hence we obtain on $\mathbb{CP}_1 \setminus \Lambda$ a well-defined map $u(x)$ inverse to $x(u)$, normalized by the condition $u(\infty) = u_\infty$, and inheriting the mirror symmetry $\overline{u}(x) = u(\overline{x})$. This map blows up the cuts in the labyrinth to smoothly embedded circles symmetrically threaded on the real axis in the same order as the cuts $\Lambda_0, \Lambda_1, \dots, \Lambda_g$. Hence

$$0 = \mathbb{R} \cap (u\Lambda_0) < \mathbb{R} \cap (u\Lambda_1) < \dots < \mathbb{R} \cap (u\Lambda_g) < u_\infty. \tag{22}$$

Each set $\mathbb{R} \cap (u\Lambda_i)$, $i = 1, \dots, g$, consists of two points, mapped one into another by $G_i(u)$ if $i \in \mathfrak{i}$ or fixed by it if $i \in \mathfrak{i}'$. The circle C_i with centre on $\mathbb{R}$ passing through the points $\mathbb{R} \cap (u\Lambda_i)$ contains the fixed points of the rotation $G_i(u)$ and by inequality (22) is disjoint from the other circles of this kind. We see that the system of generators $G_0, G_1, \dots, G_g$ defines an element of the deformation space $\mathcal{G}_g^k(\mathfrak{i})$.

The maps in parts (1) and (2) of the proof are inverse to each other. The labyrinth $\{\Lambda_s\}_{s=0}^g$ and the labyrinth $\{xC_s\}_{s=0}^g$ obtained from it by means of the composite $\mathcal{L}_g^k(\mathfrak{i}) \to \mathcal{G}_g^k(\mathfrak{i}) \to \mathcal{L}_g^k(\mathfrak{i})$ have the same representation χ_Λ by construction. Hence by Theorem 2 they belong to the same class of the labyrinth space. The bijection $\mathcal{L}_g^k(\mathfrak{i}) \leftrightarrow \mathcal{G}_g^k(\mathfrak{i})$ just constructed is continuous by Theorem 5, since the local coordinate variables in the labyrinth space were borrowed from the moduli space $\mathcal{H}_g^k$.

§ 3. Calculations in the moduli space

An effective calculation of extremal polynomials requires, first of all, the solution of Abel's equations (5) defined on the universal cover of the moduli space. These equations have been thoroughly studied in [1]; we present here only a brief survey.

On each curve $M(\mathbf{e})$ of the moduli space there exists a unique differential of the third kind $d\eta_M$ with purely imaginary periods and simple poles at infinity such that

$\operatorname{Res} d\eta_M|_{\infty_\pm} = \mp 1$. This differential is real: $\overline{J} d\eta_M = \overline{d\eta_M}$, therefore the integrals of $d\eta_M$ over the even cycles $C^+ := \overline{J}C^+$ on M vanish. The integrals of $d\eta_M$ over the odd cycles $C^- := -\overline{J}C^-$ define locally the period map onto $\mathcal{H}_g^k$. As usual, the moduli space is multiply connected and the period map cannot be continued to a global one, since going along a non-trivial cycle in $\mathcal{H}_g^k$ results in a change of the basis in the odd homology lattice $H_1^-(M,\mathbb{Z}) := \{C \in H_1(M,\mathbb{Z}) : C = -\overline{J}C\}$. This problem can be eliminated by a transition to the universal cover of the moduli space. In its labyrinth model $\mathcal{L}_g^k(\mathfrak{i})$ each element $(\mathbf{e}, \Lambda)$ possesses a distinguished basis in the lattice $H_1^-(M,\mathbb{Z})$. Namely, the cycle C_s^- corresponds to counterclockwise motion along the bank of the cut Λ_s on the upper leaf $M(\mathbf{e})$. The left-hand sides of the equalities

$$-i\int_{C_s^-} d\eta_M = 2\pi\frac{m_s}{n}, \qquad s = 0, 1, \dots, g, \quad m_s \in \begin{cases} \mathbb{Z}, & s \in \mathfrak{i}', \\ 2\mathbb{Z}, & s \in \mathfrak{i}, \end{cases} \tag{23}$$

define the *period map* $\Pi\colon \mathcal{L}_g^k(\mathfrak{i}) \to \mathbb{R}^{g+1}$, whose values lie in a hyperplane: the integral of $d\eta_M$ over the cycle $C_0^- + C_1^- + \dots + C_g^-$ is always $2\pi i$. The period map is a submersion in $\mathbb{R}^g$ [1] with a known range [3].

The points M of the moduli space associated with real polynomials of degree n fill real analytic submanifolds of dimension g that are the inverse image, under the period map, of the lattice defined by the right-hand side of equations (23). These equations are equivalent to the existence on M of a real meromorphic (Akhiezer) function with divisor $n(\infty_- - \infty_+)$:

$$\widetilde{P}_n(x,w) := \exp\biggl(n\int_{(e,0)}^{(x,w)} d\eta_M\biggr). \tag{24}$$

Its composite with the Zhukovskiĭ function is the extremal polynomial:

$$P_n(x) = \frac{1}{2}\biggl(\widetilde{P}_n(x,w) + \frac{1}{\widetilde{P}_n(x,w)}\biggr). \tag{25}$$

For an effective solution of Abel's equations (23) and the subsequent recovery of the polynomial by formulae (24), (25) we uniformize the curves $M \in \mathcal{H}_g^k$ by the Schottky groups $\mathfrak{S}$ generated by elements of the deformation space $\mathcal{G}_g^k(\mathfrak{i})$ with appropriate set $\mathfrak{i}$. As is known [10], [4], [11], the linear theta-series of Poincaré of such groups converge absolutely and uniformly on compact subsets of the discontinuity domain $\mathcal{D}$. Summing these series one obtains Abelian differentials on curves and, in particular, $d\eta_M$. After identifying the labyrinth space $\mathcal{L}_g^k(\mathfrak{i})$ and the deformation space $\mathcal{G}_g^k(\mathfrak{i})$ of the special Kleinian groups the cycles $C_1^-, C_2^-, \dots, C_g^-$ related to the labyrinth are taken to the circles $C_1, C_2, \dots, C_g$ bounding the fundamental domain of the group and the poles ∞_+, ∞_- of the differential $d\eta_M$ are taken to the points u_∞ and $G_0 u_\infty$, respectively. Recall that $u_\infty = 1$ for $0 \in \mathfrak{i}'$ and $u_\infty = \infty$ for $0 \in \mathfrak{i}$.

Abel's equations (23) and the Chebyshev representation (24), (25) can be written also in terms of the global coordinate variables $(c_s, r_s)_{s=1}^g$ in the space $\mathcal{G}_g^k(\mathfrak{i})$.

Thereupon one comes across the problem of *navigation* in the moduli space. *From an arbitrary point* $\mathcal{G}_g^k(\mathfrak{i})$ *one must descend to the smooth submanifold described by Abel's equations and moving along it, find the curve* M *corresponding to the polynomial* $P_n(x)$ *with prescribed constraints.* The variational formulae of § 3.2 enable one to obtain a local solution of the navigation problem.

3.1. Function theory in the Schottky model. One obtains an Abelian differential of the 3rd kind $d\eta_{zy}$ with poles at points z and y in the fundamental domain $\mathcal{R}(\mathfrak{S})$ by averaging over the group $\mathfrak{S}$ of the differential on the sphere [11]:

$$d\eta_{zy}(u) := \sum_{S\in\mathfrak{S}} \left\{ \frac{1}{Su-z} - \frac{1}{Su-y} \right\} dS(u) = \sum_{S\in\mathfrak{S}} \left\{ \frac{1}{u-Sz} - \frac{1}{u-Sy} \right\} du; \quad (26)$$

the two sums are termwise equal in view of the infinitesimal form of the cross ratio identity. Differentiating (26) with respect to the position of the pole z we obtain Abelian differentials of the second kind:

$$d\omega_{mz}(u) := D_z^m d\eta_{zy}(u) = m! \sum_{S\in\mathfrak{S}} (Su-z)^{-m-1}\, dS(u), \qquad m = 1, 2, \dots. \quad (27)$$

One obtains a holomorphic differential by placing the poles z and y in the same orbit of the group $\mathfrak{S}$ and isolating in (26) a telescopic sum:

$$\begin{aligned} d\zeta_j(u) := d\eta_{S_j yy} &= \sum_{S\in\mathfrak{S}|\langle S_j\rangle} \left\{ (u-S\alpha_j)^{-1} - (u-S\beta_j)^{-1} \right\} du \\ &= \sum_{S\in\langle S_j\rangle|\mathfrak{S}} \left\{ (Su-\alpha_j)^{-1} - (Su-\beta_j)^{-1} \right\} dS(u), \quad j = 1, \dots, g; \end{aligned} \quad (28)$$

summation proceeds over representatives of cosets by the subgroup $\langle S_j\rangle$ of $\mathfrak{S}$ generated by the element S_j; α_j and β_j are the attracting and the repelling fixed points of S_j, respectively. Integrating the series (26) and (27) termwise over the circles $\{C_s\}_{s=1}^g$ we find the normalization of the differential under consideration:

$$\int_{C_s} d\eta_{zy} = 0, \quad \int_{C_s} d\omega_{mz} = 0, \quad \int_{C_s} d\zeta_j = 2\pi i \delta_{sj}, \quad z, y \in \mathcal{R}(\mathfrak{S}), \ \ s, j = 1, \dots, g. \quad (29)$$

The so-called Schottky functions [7], [11], the exponentials of the integrals of the series (26) and (28), can be effectively calculated

$$(u, v; z, y) := \exp \int_v^u d\eta_{zy} = \prod_{S\in\mathfrak{S}} \frac{u-Sz}{u-Sy} : \frac{v-Sz}{v-Sy}, \quad (30)$$

$$E_j(u) := \exp \int_\infty^u d\zeta_j = \prod_{S\in\mathfrak{S}|\langle S_j\rangle} \frac{u-S\alpha_j}{u-S\beta_j}, \quad j = 1, \dots, g, \quad (31)$$

and are transformed by well-known formulae under the action of $\mathfrak{S}$:

$$(S_j u, v; z, y) = (u, v; z, y) \frac{E_j(z)}{E_j(y)}, \quad (32)$$

$$E_s(S_j u) = E_s(u) E_{sj}; \quad (33)$$

E_{lj}, the exponential of the period of the holomorphic differential, has the representation

$$E_{lj} = E_{jl} = \prod_{S\in\langle S_l\rangle|\mathfrak{S}|\langle S_j\rangle} \frac{S\alpha_j - \alpha_l}{S\alpha_j - \beta_l} : \frac{S\beta_j - \alpha_l}{S\beta_j - \beta_l}, \quad l, j = 1, \dots, g, \tag{34}$$

here we take the product over two-sided cosets in the group $\mathfrak{S}$ and for $j = l$ the coefficient $0/\infty$ corresponding to $S = 1$ is replaced by the dilation coefficient $\lambda_l := \dot{S}_l(\alpha_l)$.

A non-trivial meromorphic function on the orbit manifold of the group $\mathfrak{S}$ can be expressed in terms of the Schottky function.

Lemma 9. *Let $F(u)$ be an automorphic function with divisor $\sum_{s=1}^{\deg F}(z_s - y_s)$ in the fundamental domain of the group $\mathfrak{S}$. Then the following representation holds*:

$$F(u) = \mathrm{const} \prod_{s=1}^{\deg F} (u, *; z_s, y_s) \prod_{s=1}^{g} E_s^{m_s}(u), \tag{35}$$

where $m_s \in \mathbb{Z}$ is the increment of $(2\pi i)^{-1}\log F(u)$ over the cycle C_s. The derivatives of the automorphic function $F(u)$ with respect to the independent variable are recursively calculated by the formula

$$D_u^{m+1}F(u) = \sum_{l=0}^{m} \binom{l}{m} \cdot \left(D_u^{m-l}F(u)\right) \cdot D_u^l \left(\sum_{s=1}^{\deg F} \dot{\eta}_{z_s y_s}(u) + \sum_{s=1}^{g} m_s \dot{\zeta}_s(u) \right). \tag{36}$$

The series in $\dot{\eta}_{zy}(u) := \dfrac{d\eta_{zy}(u)}{du}$ and $\dot{\zeta}_s(u) := \dfrac{d\zeta_s(u)}{du}$ for the D_u^l are absolutely convergent.

Remark. The constraints imposed by Abel's theorem on the divisor of F are precisely the conditions for the automorphy of the right-hand side of (35).

Proof. We expand the differential dF/F in a sum of third-kind differentials and holomorphic differentials:

$$\frac{dF}{F} = \sum_{s=1}^{\deg F} d\eta_{z_s y_s} + \sum_{s=1}^{g} m_s\, d\zeta_s.$$

Integrating to u and exponentiating we arrive at (35). Differentiating repeatedly the composite function

$$F(u) = \exp\left(\int_*^u \frac{dF}{F} \right)$$

by the binomial formula we obtain (36). Effective expressions for the derivatives of the differentials in the last formula can be derived from Riemann's relations

$$\dot{\eta}_{zy}(u) = D_u \int_y^z d\eta_{uw} = \int_y^z d\omega_{1u}, \qquad \dot{\zeta}_j(u) = \int_w^{S_j w} d\omega_{1u} \quad \text{for all } w \in \mathcal{D}(\mathfrak{S}).$$

Differentiating the required number of times with respect to the parameter under the integral sign and integrating termwise we obtain series absolutely convergent in $\mathcal{D}(\mathfrak{S})$:

$$D_u^l \dot{\eta}_{zy}(u) = l! \sum_{S\in\mathfrak{S}} \bigl\{(Sy-u)^{-l-1} - (Sz-u)^{-l-1}\bigr\}, \tag{37}$$

$$D_u^l \dot{\zeta}_j(u) = l! \sum_{S\in\mathfrak{S}\,|\,\langle S_j\rangle} \bigl\{(S\beta_j-u)^{-l-1} - (S\alpha_j-u)^{-l-1}\bigr\}. \tag{38}$$

3.2. Variations of Abelian integrals. The Abelian integrals in the prescribed limits and their periods are functions of the point in the deformation space $\mathcal{G}_g^k(\mathfrak{i})$. For instance, the expressions

$$\int_v^w d\eta_{zy}, \quad \int_v^w d\omega_{mz}, \quad \int_{S_j w}^w d\omega_{mz}, \quad \int_v^w d\zeta_s, \quad \int_{S_j w}^w d\zeta_s, \quad s,j=1,2,\dots,g, \tag{39}$$

with fixed points $z, y,\ v, w$ in the fundamental domain of a Schottky group $\mathfrak{S}$ depend on the modules $\{c_s, r_s\}_{s=1}^g$. A small perturbation $\{\delta c_s, \delta r_s\}_{s=1}^g$ of the moduli results in small perturbations of the matrices $\widehat{G}_s \in \mathrm{PGL}_2(\mathbb{R})$ corresponding to the generators of the group $\mathfrak{G}$:

$$\widehat{G}_s := \begin{Vmatrix} c_s & \pm r_s^2 - c_s^2 \\ 1 & -c_s \end{Vmatrix}, \ \delta\widehat{G}_s := \begin{Vmatrix} 1 & -2c_s \\ 0 & -1 \end{Vmatrix} \delta c_s \pm \begin{Vmatrix} 0 & 2r_s \\ 0 & 0 \end{Vmatrix} \delta r_s + o, \ s=1,\dots,g; \tag{40}$$

the sign $\pm$ depends on the one of the sets, $\mathfrak{i}$ or $\mathfrak{i}'$, containing the index s;

$$o := o\biggl(\sum_{s=1}^g |\delta c_s| + |\delta r_s|\biggr).$$

Theorem 7. *The variations of the functions* (39) *are described by the formulae*

$$\delta \int_v^w d\eta = \frac{1}{2\pi i} \sum_{s=1}^g \int_{C_s} \dot{\eta}(u)\dot{\eta}_{vw}(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat{G}_s\cdot\widehat{G}_s^{-1}]\,du + o, \tag{41}$$

$$\delta \int_{S_j w}^w d\eta = \frac{1}{2\pi i} \sum_{s=1}^g \int_{C_s} \dot{\eta}(u)\dot{\zeta}_j(u)\,\mathrm{tr}[\mathcal{M}(u)\cdot\delta\widehat{G}_s\cdot\widehat{G}_s^{-1}]\,du + o; \tag{42}$$

all the objects on the right-hand sides of these equalities relate to the unperturbed group, $o := o\bigl(\sum_{s=1}^g |\delta c_s| + |\delta r_s|\bigr)$, $d\eta(u) := \dot{\eta}(u)\,du$ *is one of the differentials* $d\eta_{zy}$, $d\zeta_s$, *and* $d\omega_{mz}$, *and* $\mathcal{M}(u) := (u,1)^t\cdot(-1,u) \in \mathrm{sl}_2(\mathbb{C})$ *is the Hejhal matrix.*

Proof. In the special case $\mathfrak{i} = \varnothing$ the proof is presented in [4]; however, it can be literally transferred to the case of general Schottky groups.

Remark. The use of quadrature formulae for the calculation of the right-hand sides in (41) and (42) is inefficient because it requires summation of Poincaré series at many points. A trick allowing one to calculate these integrals by summing series only at $2g-1$ points is described in [4].

3.3. Parametric calculation of polynomials. For an illustration of Lemma 9 we find an effectively calculated parametric representation of extremal polynomials under the assumption that Abel's equations are satisfied.

The circles $C_1, C_2, \dots, C_g$ make up half the canonical basis of 1-cycles on the compact curve $M_c := \mathcal{D}(\mathfrak{S})/\mathfrak{S}$. Hence [12] Abelian differentials on the curve can be normalized by a prescription of their periods along these circles. In particular, the differential $d\eta_M$ associated with the curve M with periods prescribed by Abel's equations (23) has the representation

$$d\eta := d\eta_{zy} + \sum_{s=1}^{g} \frac{m_s}{n}\, d\zeta_s, \qquad z := G_0 u_\infty, \quad y := u_\infty. \tag{43}$$

It is now easy to obtain an expression for the Akhiezer function $\widetilde{P}_n(u)$ from which one can recover the extremal polynomial P_n by formula (25). In a similar way one finds the independent variable $x(u)$ defined in general up to affine motions. The results of our calculations are collected in Table 1, in which we take into account the form of the generator $G_0(u)$.

Table 1

	$0 \in \mathfrak{i}'$	$0 \in \mathfrak{i}$
Normalization of $x(u)$	$u = (0, 1, \infty) \to x = (0, \infty, -1)$	$u = (\pm i, \infty) \to x = (\pm i, \infty)$
$x(u) =$	$-(u, \infty; 0, 1)(u, \infty; 0, -1)$	$i\dfrac{x_1 + 1}{x_1 - 1}$, $x_1(u) := (u, \infty; -i, i)^2$
$\widetilde{P}_n(u) =$	$(u, \infty; -1, 1)^n \prod_{j=1}^{g} E_j^{m_j}(u)$	$(u, i; 0, \infty)^n \prod_{j=1}^{g} \left(\dfrac{E_j(u)}{E_j(i)}\right)^{m_j}$

Using formula (36) one can calculate the jets of the functions $\widetilde{P}_n(u)$, $x(u)$, and therefore the derivatives $D_x^m P_n(x)$, $m = 0, 1, 2, \dots$, of the extremal polynomial. In terms of the values of these derivatives at various points one can express the constraints of the optimization problem: for instance, the two leading coefficients of the polynomial are $P_n(x)x^{-n}$ and $(nP_n(x) - x\dot{P}_n(x))x^{1-n}$ for $u = u_\infty$.

3.4. Abel's equations in the space $\mathfrak{G}_g^k(\mathfrak{i})$. Of course a meromorphic function with divisor $n(\infty_- - \infty_+)$ does not exist on each curve M. Conditions for the automorphy of the function $\widetilde{P}_n(u)$ in Table 1 are equivalent to Abel's equations (23).

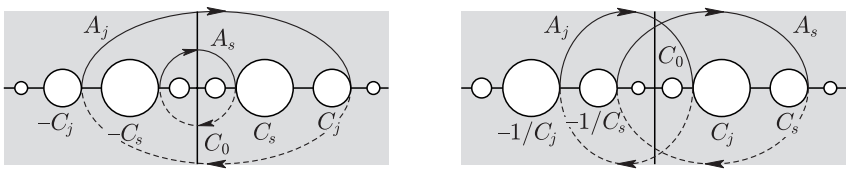


Figure 5. Calculation of $A_* - \overline{J}A_*$, $s \in \mathfrak{i}'$, $j \in \mathfrak{i}$: (a) for $0 \in \mathfrak{i}'$; (b) for $0 \in \mathfrak{i}$

Lemma 10. *Abel's equations* (23) *are equivalent to the g real relations*:

$0 \in \mathfrak{i}'$	$0 \in \mathfrak{i}$
$E_s^{2n}(1) = E_{1s}^{m_1} E_{2s}^{m_2} \cdots E_{gs}^{m_g}$, $s = 1, \dots, g$	$E_s^n(0) E_{1s}^{m_1} E_{2s}^{m_2} \cdots E_{gs}^{m_g} = 1$, $s = 1, \dots, g$

Proof. We can formulate Abel's equations as follows: *the differential* (43) *normalized by conditions* (23) *must have purely imaginary periods on the curve.* This condition holds on the cycles $C_1, C_2, \dots, C_g$ in view of the normalization (29) of the holomorphic differentials. Let A_j, $j = 1, \dots, g$, be an arc in $\mathcal{R}(\mathfrak{S}) \cap \mathbb{H}$ joining the real point $u \in G_0 C_j$ and $S_j u \in C_j$. Using the intersection form one can verify that the $2g$ cycles $C_1, \dots, C_g$; $A_1, \dots, A_g$ form a basis in the lattice of integral 1-cycles on the compact curve $M_c := \mathcal{D}(\mathfrak{S})/\mathfrak{S}$. We see from Fig. 5 that

$$A_j - \overline{J} A_j = \chi(\mathfrak{i}, 0) \cdot C_0^- + \chi(\mathfrak{i}, j) \cdot C_j^- \pmod{2H_1^-(M, \mathbb{Z})}, \tag{44}$$

where $\chi(\mathfrak{i}, \cdot)$ is the characteristic function of the set $\mathfrak{i}$ taking values 0 and 1. Since the differential $d\eta$ is real, taking account of normalization conditions (29), the fact that the index m_j is even for $j \in \mathfrak{i}$, and the congruence (44) we obtain $\operatorname{Im} \int_{A_j} n \, d\eta \in 2\pi i \mathbb{Z}$. Hence the periods of $d\eta$ are purely imaginary if and only if $\exp\left(\int_{A_j} n \, d\eta\right) = 1$, $j = 1, \dots, g$. The transformation rules (32), (33) for the Schottky functions transform the remaining g relations into the form required in the statement of the theorem.

3.5. Scheme of the algorithm. We now describe a protocol for the solution of least deviation problems in the framework of our approach.

(1) Given the problem data, find the topological invariants g, k and the integer indices $m_0, m_1, \dots, m_g$ corresponding to the solution $P_n(x)$. This is related to finding a low-dimensional face of the sphere $\{Q_n(x) : \|Q_n\|_E = \text{const}\}$ in the space of polynomials containing the solution $P_n(x)$. The author knows of no algorithm implementing this part of the protocol. The integer indices $m_0, m_1, \dots, m_g$ can be guessed; sometimes one knows their asymptotic values as $n \to \infty$, for instance, in the problem of the least deviation of a monic polynomial on several intervals of the real axis.

(2) Fix a partitioning of the index set $\{0, 1, \dots, g\} = \mathfrak{i} \cup \mathfrak{i}'$. This produces a realization of the universal covering space $\widetilde{\mathcal{H}}_g^k$ as a subdomain of the Euclidean space explicitly defined by the system of inequalities (7), (8).

(3) Make a descent from an arbitrary point in the moduli space onto the smooth g-dimensional submanifold $\mathbb{T}$ of the domain $\mathcal{G}_g^k(\mathfrak{i})$ described by Abel's equations in Lemma 10. Locally, navigation in the moduli space is performed with the help of variational formulae (41), (42) enabling one to implement Newton's or other descent methods.

(4) Using formulae (36) for derivatives of the automorphic functions and variational formulae for Abelian integrals find on $\mathbb{T}$ a point M with polynomial satisfying the constraints of the least deviation problem.

(5) Recover the solution $P_n(x)$ from the associated curve M using the parametric formulae in Table 1.

We plot the graphs of several extremal polynomials calculated by means of software realizing parts (3)–(5) of the protocol in Fig. 6.

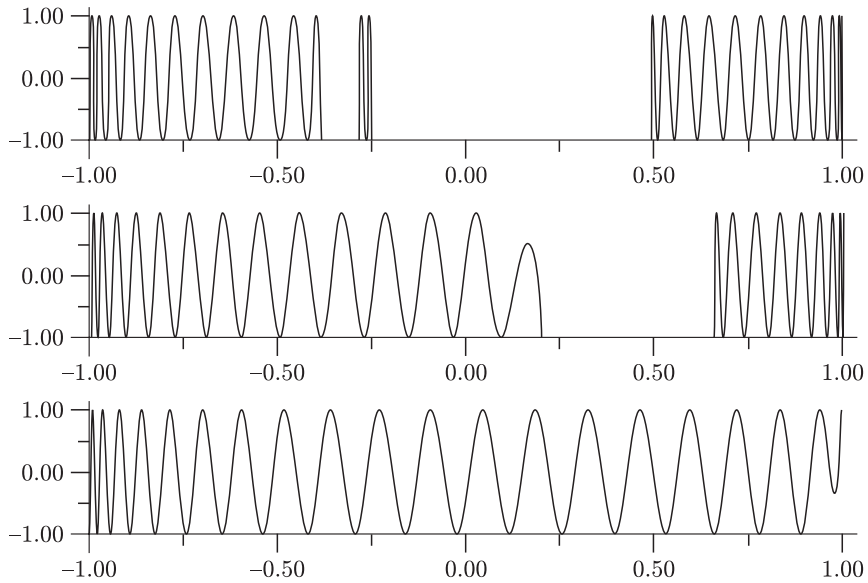


Figure 6. The extremal polynomials $P_{50}(x)$ for $g = 2$, $k = 3, 2, 1$

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