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# Chebyshev representation for rational functions

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**Abstract.** An effective representation is obtained for rational functions all of whose critical points, apart from $g-1$, are simple and their corresponding critical values lie in a four-element set. Such functions are described using hyperelliptic curves of genus $g \geqslant 1$. The classical Zolotarëv fraction arises in this framework for $g = 1$.

Bibliography: 8 titles.

**Keywords:** rational approximation, Zolotarëv fraction, Riemann surfaces, Abelian integrals.

## § 1. Introduction

The following optimization problem arises in the natural way in the construction of multiband digital and analogue filters.

Let $E$ be a fixed system of disjoint closed intervals on the real axis. On each interval a pulse function $F(x)$ is set equal to 0 (stopband) or 1 (passband). Among all real rational functions $R(x)$ of fixed degree $n$ find the best approximation to the pulse function in the uniform metric on the intervals under consideration:

$$\|R - F\|_E := \max_{x \in E} |R(x) - F(x)| \longrightarrow \min.$$

The solution of this problem has $2n+2$ alternance points (see [1]), that is, points at which the deviation attains its norm (with alternating sign). Each alternance point lying in the interior of $E$ is a critical point of the solution; it corresponds to a critical value in the set of 4 elements $\pm\|R - F\|_E$, $1 \pm \|R - F\|_E$. When $E$ consists of only a few intervals, the number of interior alternance points is almost equal to $2n - 2$, the total number of critical points of the solution counted with multiplicities.

The aim of this paper is to find an effective representation for rational functions the great majority of whose critical points are simple and correspond to four distinct critical values. A similar problem arises in the investigation of the problem of least deviation in the uniform norm for polynomials. Chebyshev invented a method for describing polynomials the great majority of whose critical points are simple, with the corresponding critical values belonging to a 2-element set. We will show how

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Chebyshev's construction can be modified to describe rational functions with four exceptional critical values.

I dedicate this paper to the memory of V. I. Lebedev and F. Peherstorfer.

## § 2. Chebyshev representation for polynomials

We will give an overview of the Chebyshev construction for polynomials; for a detailed presentation the reader can consult [2] and [3]. Without loss of generality we assume that the two distinguished critical values are $\pm1$. Informally speaking, the extremality number of a polynomial is $g$ if all the critical points of the polynomial, apart from $g$ of them, are simple and correspond to the values $\pm1$. Here 'extremality' has two roles: first, we distinguish the critical points which do not meet a certain general requirement; second, polynomials with small $g = 0, 1, 2, \dots$ are more likely to provide solutions to various extremal problems involving the uniform norm.

**2.1. The extremality number of a polynomial.** By the *extremality number* of a polynomial $P(x)$ we mean

$$g = \sum_{x:P(x)\neq\pm1} \operatorname{ord} P'(x) + \sum_{x:P(x)=\pm1} \left[\frac{1}{2}\operatorname{ord} P'(x)\right], \tag{2.1}$$

where $\operatorname{ord} P'(x)$ is the order of the zero of the derivative of the polynomial $P$ at $x \in \mathbb{C}$ and $[\,\cdot\,]$ is the integer part of a number.

**2.2. The Riemann surface.** In the Chebyshev framework, corresponding to a polynomial $P(x)$ there is a Riemann surface $M = M(\mathsf{E})$ branched over the points $\mathsf{E} := \{e_s\}_{s=1}^{2g+2}$ at which the polynomial takes values $\pm1$ with odd multiplicity (so that generically these are simple values). The affine part of the curve is described by

$$w^2 = \prod_{s=1}^{2g+2} (x - e_s), \qquad (x, w) \in \mathbb{C}^2. \tag{2.2}$$

We can show (see [2] and [3]) that the extremality number (2.1) of a polynomial is equal to the genus of the associated Riemann surface $M$. There are effective methods to determine whether a fixed Riemann surface (2.2) is associated with a polynomial. On $M$ there is a unique Abelian differential of the third kind $d\eta_M$ with simple poles at infinity, residues $\pm1$ and purely imaginary periods. The curve (2.2) is associated with a polynomial of degree $n$ if and only if

$$\int_C d\eta_M \in \frac{2\pi i}{n}\mathbb{Z} \qquad \forall\, C \in H_1(M, \mathbb{Z}). \tag{2.3}$$

The above inclusions impose $2g$ transcendental real conditions on the $2g-1$ complex moduli of the curve $M(\mathsf{E})$. If the curve is real (for instance, it is associated with a real polynomial), then half these conditions are automatically satisfied, thanks to mirror symmetry, and we obtain $g$ real constraints on the real moduli. The polynomials generating the curve can be recovered by the explicit formula

$$P(x) = \pm\cos\left(ni \int_{(e,0)}^{(x,w)} d\eta_M\right), \qquad e \in \mathsf{E}, \tag{2.4}$$

where the result of the calculation does not depend on the path of integration on $M$, the two ways to choose $w(x)$ or the branch point $e$ taken as the initial point of integration. For practical purposes it is also significant that the complexity of the computation in (2.4) does not depend on the degree $n$ of the polynomials. In the special cases of $g = 0$ and $g = 1$ this formula yields the classical Chebyshev and Zolotarëv polynomials, respectively. We shall call (2.4) the *Chebyshev representation of a polynomial*. The motivation behind this representation comes from the classical Pell-Abel equation and its geometric interpretation.

**2.3. The motivation behind the Chebyshev correspondence.** Let $P(x)$ be a polynomial of degree $n$. Pulling out all the squared factors in $P^2-1$, the Pell-Abel equation

$$P^2(x) - D(x)Q^2(x) = 1$$

holds, where the polynomial $D(x)$ is free from squares and has leading coefficient 1. The zeros of the coefficient $D(x)$ of the Pell-Abel equation form the branch divisor $\mathsf{E}$ of the curve (2.2) associated with $P(x)$. It is easy to see that $Q(x)$ divides $dP(x)/dx$, so that for some polynomial $\rho(x) = x^g + \cdots$ we have

$$P'(x) = n\rho(x)Q(x).$$

As a result, the Pell-Abel equation turns into the differential equation

$$\frac{dP(x)}{\sqrt{P^2(x) - 1}} = n\frac{\rho(x)\,dx}{\sqrt{D(x)}},$$

which can readily be solved:

$$P(x) = \pm\cos\biggl(ni\int_e^x \frac{\rho(s)\,ds}{\sqrt{D(s)}}\biggr).$$

Of course, if $D(x)$ and $\rho(x)$ are arbitrary polynomials, then the right-hand side of the above formula is not a polynomial. To ensure that it is a single-valued function of $x$ the following inclusions are necessary and sufficient:

$$\int_C \frac{\rho(x)\,dx}{\sqrt{D(x)}} \in \frac{2\pi i}{n}\mathbb{Z}$$

for all integer 1-cycles $C$ on the Riemann surface $M(\mathsf{E})$. In particular, the differential $\dfrac{\rho(x)dx}{\sqrt{D(x)}}$ has only purely imaginary periods, so it coincides with the differential $d\eta_M$ completely determined by the Riemann surface.

*Remark* 1. On the way we have obtained the following necessary and sufficient solvability condition for the Pell-Abel equation (as regards some other conditions, see [4] and [5]). On the curve $w^2 = D(x)$ determined by the coefficient $D(x)$ of the Pell-Abel equation, the particularly normalized Abelian differential $\frac{1}{2\pi i}\,d\eta_M$ must have rational periods. If this holds, then the solutions $P$ and $Q$ have the following form:

$$P(x) = \pm\cosh\biggl(n\int_{(e,0)}^{(x,w)} d\eta_M\biggr), \qquad \deg P = n,$$

$$Q(x)w(x) = \pm\sinh\biggl(n\int_{(e,0)}^{(x,w)} d\eta_M\biggr), \qquad \deg Q = n-g-1,$$

where the integer $n$ is a multiple of the denominators of all the periods. This representation implies the following structure of solutions $P(x)$ of the Pell-Abel equation: there exists a unique solution $P_0(x)$ (up to sign) of lowest positive degree; all the other solutions are composites of this prime solution with a classical Chebyshev polynomial.

## § 3. The Chebyshev representation for rational functions

By a suitable linear fractional transformation we can take a 4-tuple of points in the Riemann sphere to a set

$$\mathsf{Q} = \mathsf{Q}(\varkappa) := \left\{-1, 1, -\frac{1}{\varkappa}, \frac{1}{\varkappa}\right\}, \qquad \varkappa \in \mathbb{C} \setminus \{0, \pm 1\}.$$

Roughly speaking, the extremality number of a rational function is equal to $g$ if all its critical points apart from $g-1$ are simple and correspond to values in $\mathsf{Q}$. Of course, the extremality number depends on the modulus $\varkappa$, but throughout this paper the set $\mathsf{Q}$ will be fixed, so this dependence will not be explicitly indicated.

**3.1. The extremality number of a rational function.** The extremality number of a rational function $R(x)$ with respect to a set of values $\mathsf{Q}$ is

$$g = 1 + \sum_{x : R(x) \notin \mathsf{Q}} \operatorname{ord} dR(x) + \sum_{x : R(x) \in \mathsf{Q}} \left[\frac{1}{2} \operatorname{ord} dR(x)\right], \tag{3.1}$$

where the sum is taken over all points in the Riemann sphere; $\operatorname{ord} dR(x)$ is the order of the zero at the point $x$ of the differential of the holomorphic map $R \colon \mathbb{C}P^1 \to \mathbb{C}P^1$ (for instance, at simple poles of $R(x)$ this quantity is equal to zero).

To describe rational functions that take one of the four prescribed values at most critical points we take the Chebyshev approach.

**3.2. The Riemann surface.** With the rational function $R(x)$ we associate the two-sheeted Riemann surface $M = M(\mathsf{E})$ branched over the points $\mathsf{E} := \{e_s\}_{s=1}^{2g+2}$, at which the function takes values in $\mathsf{Q}(\varkappa)$ with odd multiplicity (that is, generically the values are simple). The affine part of the curve is described by

$$w^2 = \prod_{s=1}^{2g+2} (x - e_s), \qquad (x, w) \in \mathbb{C}^2. \tag{3.2}$$

If a branch point lies at infinity, then there is no corresponding factor in the equation of the curve.

**Lemma 1.** *The genus of a Riemann surface $M(\mathsf{E})$ associated with the rational function $R(x)$ is equal to the extremality number of this rational function.*

*Proof.* A rational function $R(x)$ of degree $n$ has $2n-2$ critical points (counting multiplicities) on the Riemann sphere. Taking account of the definition (3.1) of the extremality number and the identity

$$\left[\frac{a}{2}\right] + \left[\frac{a+1}{2}\right] = a, \qquad a \in \mathbb{Z},$$

we can write

$$2n-2=\sum_{x}\operatorname{ord} dR(x)=g-1+\sum_{x:R(x)\in\mathsf{Q}}\left[\frac{1}{2}(\operatorname{ord} dR(x)+1)\right].$$

The total number of points in $R^{-1}\mathsf{Q}$ with multiplicities taken into account is $4n$, and the number of points with odd multiplicities is by definition equal to the number of branch points of the curve $M(\mathsf{E})$:

$$4n=\sum_{x:R(x)\in\mathsf{Q}}(\operatorname{ord} dR(x)+1)=\deg\mathsf{E}+\sum_{x:R(x)\in\mathsf{Q}}2\left[\frac{1}{2}(\operatorname{ord} dR(x)+1)\right];$$

in the last equality we have used the fact that $2[\frac{a}{2}]$ is equal to $a$ if $a$ is even and to $a-1$ if $a$ is odd. Comparing the last two equalities we obtain $\deg\mathsf{E}=2g+2$, so that the hyperelliptic curve $M(\mathsf{E})$ has genus $g$.

**3.3. Motivation in the spirit of the Pell-Abel equation.** Let $R(x)$ be a rational function represented as an irreducible fraction. We pull out all squared factors in the numerator of the left-hand side of the following expression, obtaining

$$(R^2(x)-1)(\varkappa^2R^2(x)-1)=:D_{2g+2}(x)S^2(x), \tag{3.3}$$

where $S(x)$ is a rational function with denominator equal to the square of that of $R(x)$. The monic polynomial $D(x)$ is as on the right-hand side of (3.2). It has degree $2g+2$ if $R(x)$ does not take a value in $\mathsf{Q}$ at infinity with odd multiplicity. Otherwise $D(x)$ is a polynomial of degree $2g+1$.

It is easy to see that the zeros of $S(x)$ are also zeros of the derivative of $R(x)$, therefore

$$R'(x)=S(x)\rho(x), \tag{3.4}$$

where $\rho(x)$ is a polynomial. We can find its degree by comparing the asymptotic behaviour of $R(x)$ and $S(x)$ at infinity (we must consider the cases when $R(\infty)$ is infinite or belongs to the set $\mathsf{Q}$ separately). It turns out that we always have $\deg\rho(x)<g$. Now we can write the Pell-Abel equation (3.3) as a differential equation:

$$\frac{dR}{\sqrt{(R^2(x)-1)(\varkappa^2R^2(x)-1)}}=\frac{\rho(x)\,dx}{\sqrt{D(x)}}. \tag{3.5}$$

The latter can readily be solved, giving a representation of the original rational function in the form

$$R(x)=\operatorname{sn}\biggl(\int_e^x\frac{\rho(x)\,dx}{\sqrt{D(x)}}+A(e)\biggr), \tag{3.6}$$

where we must choose the point $e$ and the phase shift $A(e)$ to be compatible, which we will do later. The right-hand side of the above equality must be a single-valued function of $x$, so the integrals of the holomorphic differential $\rho(x)\frac{dx}{w}$ on $M(\mathsf{E})$ over all the integer 1-cycles on the surface must lie in the lattice $4K(\varkappa)\mathbb{Z}+2iK'(\varkappa)\mathbb{Z}$, where $K(\varkappa)$ and $K'(\varkappa)$ are the complete elliptic integrals with modulus $\varkappa$. Of course, for such a differential to exist imposes constraints on the moduli of the Riemann surface.

We call a representation of a rational function for the form (3.6) a *Chebyshev representation*. As with polynomials, the complexity of calculations using this formula is independent of the degree of the rational function, so this substitution can be used in solving filter optimization problems. We show below that the classical Zolotarëv fraction (solving the problem of finding the best rational approximation to the signum function) can be represented in this way.

## § 4. The image of the Chebyshev correspondence

**Theorem 1.** *A hyperelliptic curve is the image of some rational function under the Chebyshev correspondence if and only if it is a* (*possibly ramified*) *cover of the torus* $T(\mathsf{Q})$ *with affine part given by*

$$s^2 = (r^2-1)(\varkappa^2r^2-1), \qquad (r,s)\in\mathbb{C}^2. \tag{4.1}$$

*Proof.* 1. If a curve $M$ is the result of applying the Chebyshev construction to the rational function $R$, then the covering of the torus is defined by the rational map

$$M \ni (x,w) \xrightarrow{\widetilde{R}} (R(x), wS(x)) = (r,s)\in T, \tag{4.2}$$

where $S(x)$ is the same function as in the Pell-Abel type equation (3.3). We see that this covering map intertwines the hyperelliptic involutions $J_M(x,w) := (x,-w)$ on the curve $M$ and $J_T(r,s) := (r,-s)$ on the torus.

We can also give a purely topological proof of the existence of the covering $M\to T$ commuting with the involutions. In the diagram

$$\begin{tikzcd} M(\mathsf{E}) \arrow[r, dashed, "\widetilde{R}"] \arrow[d, "p_M"'] & T(\mathsf{Q}) \arrow[d, "p_T"] \\ \mathbb{C}P^1 \arrow[r, "R"] & \mathbb{C}P^1 \end{tikzcd} \tag{4.3}$$

let $p_M(x,w) := x$ and $p_T(r,s) := r$ be the quotient maps of the actions of $J_M$ and $J_T$, respectively. They are 2-fold coverings branched over the points in $\mathsf{E}$ and $\mathsf{Q}$, respectively. The map $R$ can be lifted to the covering space if and only if we have the embedding of fundamental groups (see [6])

$$(R\circ p_M)\pi_1(M\setminus p_M^{-1}R^{-1}\mathsf{Q}) \subset p_T\pi_1(T\setminus p_T^{-1}\mathsf{Q}),$$

provided the base points of the fundamental groups are compatibly chosen. We shall verify this embedding on the generators of the fundamental group of the punctured surface $M$. For generators we take two kinds of loops:

(A) the $2g$ loops producing the canonical dissection of the nonpunctured surface;

(B) the loops enclosing the punctures $p_M^{-1}R^{-1}\mathsf{Q}$ on the surface and disjoint (apart from the initial point) from the loops in (A).

We can pick loops in (A) so that the projection $p_M$ of each loop is the product of two loops in $\pi_1(\mathbb{C}P^1\setminus R^{-1}\mathsf{Q})$, each enclosing a unique point in $\mathsf{E}$ (Fig. 1). The

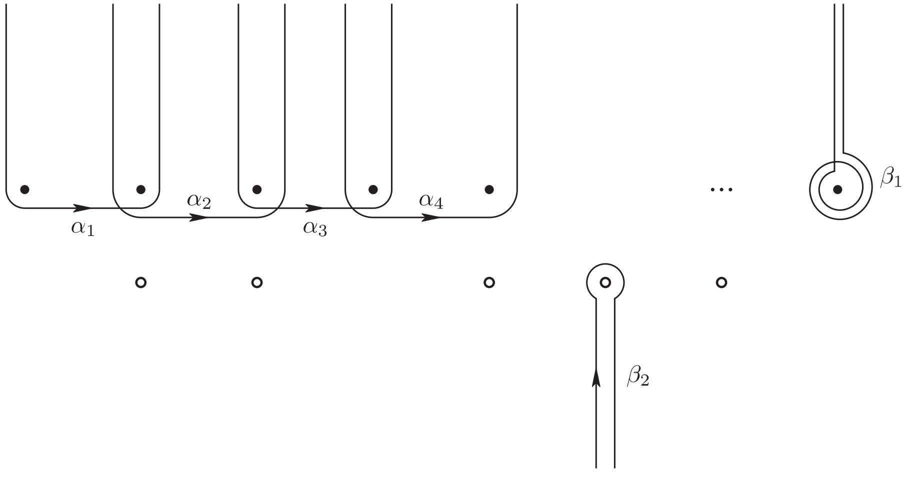


Figure 1. The projection of the generators of the group $\pi_1(M \setminus p_M^{-1} R^{-1}\mathsf{Q})$ onto the punctured sphere. Infinity is the base point of the fundamental group; $\bullet$ denotes points in the branching divisor $\mathsf{E}$; $\circ$ denotes other points in $R^{-1}\mathsf{Q}$; $\alpha_1, \alpha_2, \ldots$ are the projections of the loops in class (A); $\beta_1$ is the projection of a loop in subclass (B1) and $\beta_2$ of a loop in subclass (B2).

map $R(x)$ takes a loop of this form into a product of two loops, each making an odd number of circuits about some puncture in $\mathsf{Q}$. This product can be lifted to a loop on the torus.

A loop in class (B) either goes about a branch point $(e,0)$ — this is subclass (B1), or — for subclass (B2) — about a puncture $(x,w)$ such that $R$ has an even branching index at $x$. In either case the $Rp_M$-image of a loop in class (B) is a loop making an even number of circuits about a point in $\mathsf{Q}$. Such a loop can be lifted to a loop on the torus.

From the construction of the lifting of the map we easily deduce the equivariance already mentioned:

$$\widetilde{R} J_M = J_T \widetilde{R}. \tag{4.4}$$

2. Conversely, let a hyperelliptic curve $M$ cover the torus (4.1). We shall show below that there also exists a covering $\widetilde{R}$ intertwining the involutions $J_M$ and $J_T$. Once we have proved this, we can project the map $\widetilde{R}$ to a map $R$ between the bases in the diagram (4.3). Since we have the equivariance (4.4), the map $\widetilde{R}$ takes the fixed points $(e,0)$ of the involution $J_M$ to fixed points of the involution $J_T$ on the torus and has odd branching indices at these points. Correspondingly, the image of the branch divisor $R\mathsf{E}$ lies entirely in $\mathsf{Q}$ and the branching index of $R(x)$ at each point $e \in \mathsf{E}$ is odd. At all other points in the inverse image $x \in R^{-1}\mathsf{Q}$ this index is even because the branching indices of maps in a composition multiply together and

$$\operatorname{ind} p_M(x,w) = \begin{cases} 1, & x \notin \mathsf{E}, \\ 2, & x \in \mathsf{E}, \end{cases} \qquad \operatorname{ind} p_T(r,s) = \begin{cases} 1, & r \notin \mathsf{Q}, \\ 2, & r \in \mathsf{Q}. \end{cases} \tag{4.5}$$

We see that the Chebyshev construction assigns precisely the curve $M$ to the rational function $R(x)$.

Now we demonstrate how we can modify the fixed covering $\check{R}\colon M \to T$ to fulfill the equivariance condition. We fix a holomorphic differential $d\eta \neq 0$ on the torus. Its periods form a lattice

$$L = L(\eta) := \left\{ \int_C d\eta,\ C \in H_1(T, \mathbb{Z}) \right\}.$$

The torus can be represented as the quotient of the complex plane by this lattice, and this correspondence $T \to \mathbb{C}/L$ has the form

$$\eta(r,s) = \int_{(1,0)}^{(r,s)} d\eta \ (\mathrm{mod}\, L(\eta))$$

(changing the initial point of integration results in a translation of the torus). Now we express the action of the covering $\check{R}$ in this framework:

$$\eta(\check{R}(x,w)) - \eta(\check{R}(e,0)) = \int_{\check{R}(e,0)}^{\check{R}(x,w)} d\eta = \int_{(e,0)}^{(x,w)} d\zeta. \tag{4.6}$$

Here $d\zeta := \check{R}^* d\eta$ is the differential lifted from the torus to the curve $M$. It is holomorphic, so after the hyperelliptic involution it gets multiplied by $-1$: $J_M^*\, d\zeta = -d\zeta$. We can extend (4.6):

$$\int_{(e,0)}^{(x,w)} d\zeta = -\int_{(e,0)}^{(x,-w)} d\zeta = -\big(\eta(\check{R}(x,-w)) - \eta(\check{R}(e,0))\big). \tag{4.7}$$

Equalities (4.6) and (4.7) can be interpreted as the equivariance of $\check{R}$ with respect to the actions of the involution $J_M$ on $M$ and the involution on the torus $T$ that acts in the $\eta$-plane as

$$\eta \to 2\eta(\check{R}(e,0)) - \eta \ (\mathrm{mod}\, L).$$

This involution has 4 fixed points on the torus, $\eta(\check{R}(e,0)) + L/2 \ (\mathrm{mod}\, L)$; generally speaking it is distinct from the involution $J_T$ introduced before and acting in this model as $\eta \to -\eta \ (\mathrm{mod}\, L)$. However, combining $\check{R}$ with an appropriate translation of the torus we can define a new covering $\widetilde{R}$ satisfying the required equivariance condition (4.4). It is easy to see that there are just 4 appropriate translations of the torus. They produce 4 rational functions $\{\pm R, \pm 1/(\varkappa R)\}$, which form the orbit of the action of the Klein Vierergruppe and are associated with the same Riemann surface $M$ by means of the Chebyshev correspondence.

The proof of Theorem 1 is complete.

The image of the Chebyshev correspondence can also be described more explicitly, in the spirit of relations (2.3), which show that all the periods of a certain Abelian differential on the surface are commensurable. There will be no great loss of generality if we do this in the case of real rational functions and $\varkappa \in (0,1)$, which is important for applications. In this case $M$ is a real curve and $T$ is a real torus, that is, they carry anticonformal involutions $\bar{J}_M(x,w) := (\bar{x}, \bar{w})$ and $\bar{J}_T(r,s) := (\bar{r}, \bar{s})$. The action of these reflections imprints the structure of the 1-dimensional homology of the surface, which we discuss in the next section.

## § 5. One-dimensional cycles on a real curve

Let $M$ be a Riemann surface endowed with an anticonformal involution $\bar{J}$ (a reflection). The action of the reflection extends in a natural way to the real $2g$-dimensional first homology space of the surface, which decomposes into a sum of the subspaces corresponding to the eigenvalues $\pm 1$ of the operator $\bar{J}$:

$$H_1(M,\mathbb{R}) = H_1^+(M,\mathbb{R}) \oplus H_1^-(M,\mathbb{R}), \qquad H_1^\pm(M,\mathbb{R}) := (I \pm \bar{J})H_1(M,\mathbb{R}). \quad (5.1)$$

We shall say that cycles $C = \bar{J}C$ in the space $H_1^+(M)$ are *even*. Correspondingly, cycles $C = -\bar{J}C$ in $H_1^-(M)$ are said to be *odd*. The spaces of even and odd cycles contain the full-rank lattices of integer cycles

$$H_1^\pm(M,\mathbb{Z}) := H_1^\pm(M,\mathbb{R}) \cap H_1(M,\mathbb{Z}).$$

The space $H_1(M,\mathbb{R}) \cong \mathbb{R}^{2g}$ is endowed with a symplectic form, which is given by the intersection index of cycles. Since the reflection $\bar{J}$ reverses orientation at each point of intersection of integer cycles, it follows that

$$\bar{J}C_1 \circ \bar{J}C_2 = -C_1 \circ C_2, \qquad C_1, C_2 \in H_1(M,\mathbb{R}). \quad (5.2)$$

Hence we can easily deduce that the even and odd cycles form Lagrangian subspaces (which means that the restriction of the intersection form to these spaces is trivial); these have dimension equal to half the dimension of the ambient space.

**Lemma 2.** *The space $H_1^\pm(M,\mathbb{R})$ has dimension $g$.*

*Proof.* It follows from the decomposition (5.1) that the sum of the dimensions of the subspaces of even and odd cycles is $2g$. We claim that none of these spaces can have dimension greater than any other. Assume the converse: for instance, suppose $\dim H_1^+(M,\mathbb{R}) > \dim H_1^-(M,\mathbb{R})$. The intersection form defines a linear action of even cycles on the odd ones. In view of the inequality of dimensions, there must exist an even cycle annihilating all the odd cycles. Then it has zero intersection index with any element of $H_1(M,\mathbb{R})$, in contradiction to the nondegeneracy of the intersection form.

**Lemma 3.** 1. *A basis of the space of even* (*odd*) *cycles can be used to normalize the Abelian differentials.*

2. *The following three statements are equivalent*:

(i) *a holomorphic differential $d\xi$ is real*;

(ii) $\displaystyle\int_{C^+} d\xi \in \mathbb{R}$ *for all even cycles $C^+$*;

(iii) $\displaystyle\int_{C^-} d\xi \in i\mathbb{R}$ *for all odd cycles $C^-$*.

*Proof.* 1. Let $C_1, C_2, \dots, C_g$ be a basis in the space of even (odd) cycles. We must show that if the integrals of some holomorphic differential over the above $g$ cycles vanish, then the differential itself is trivial. We complete this basis to a basis of the full homology space. For the closed smooth 1-forms $d\omega$ and $d\xi$ on the surface $M$ and for the (not necessarily canonical or integer) homology basis $C_1, C_2, \dots, C_{2g}$

we have Riemann's bilinear relation (see [2])

$$\int_M d\omega \wedge d\xi = -\sum_{s,j=1}^{2g} F_{sj} \int_{C_s} d\omega \int_{C_j} d\xi,$$

where the matrix $F_{sj}$ is the inverse of the intersection matrix $C_s \circ C_j$. If we set $d\xi = \overline{d\omega}$ here we obtain

$$0 \leqslant \|d\omega\|^2 = i \int_M d\omega \wedge \overline{d\omega} = -i \sum_{s,j=1}^{2g} F_{sj} \int_{C_s} d\omega \overline{\int_{C_j} d\omega}. \tag{5.3}$$

If $\int_{C_j} d\omega = 0$ for $j = 1, \dots, g$, then the sum on the right-hand side only contains terms with $s, j > g$. But then $F_{sj} = 0$. Indeed, the intersection matrix has a $2 \times 2$ block structure with a zero $g \times g$ block at position $(1,1)$. The inverse matrix has a zero block of the same size at position $(2,2)$. We see that only the trivial holomorphic differential has zero periods along all the even (odd) cycles.

2. A holomorphic differential $d\xi$ is said to be *real* if the reflection takes it to its complex conjugate: $\bar{J}^* \, d\xi = \overline{d\xi}$. If $C \in H_1^{\pm}(M)$ is an even or odd cycle, then

$$\int_C d\xi = \int_{\pm \bar{J} C} d\xi = \pm \int_C \bar{J}^* \xi = \pm \int_C \overline{d\xi} = \pm \overline{\int_C d\xi}.$$

Conversely, assume that a holomorphic differential $d\xi$ has real periods along all the even cycles or has purely imaginary periods along all the odd cycles. Then the differential $\bar{J}^* \, \overline{d\xi} - d\xi$ has zero periods along all even or odd cycles, respectively. It follows from the first assertion of the lemma that this must be a trivial differential, and this is equivalent to $d\xi$ being real.

## § 6. Bases for the lattices of even and odd cycles on a hyperelliptic curve

To investigate the Chebyshev representation for real rational functions we need bases for the lattices of even and odd integer cycles on a real hyperelliptic curve, as well as bases for the sublattices of (anti)symmetrized cycles

$$L_M^{\pm} := (I \pm \bar{J}) H_1(M, \mathbb{Z}) \subset H_1^{\pm}(M, \mathbb{Z}).$$

Assume that the branch divisor $\mathsf{E}$ of the Riemann surface (3.2) contains $2k$ real points and $g-k+1$ pairs of complex conjugate points. If $k > 0$, then the topological invariant $k$ can be interpreted as the number of real ovals on the surface. The case $k = 0$ has its own peculiarities, and we consider it separately.

**6.1. The case $\boldsymbol{k > 0}$.** On the Riemann sphere we shall draw disjoint cuts which join branch points on the surface pairwise and are invariant under complex conjugation. Real branch points will be joined by the $k$ intervals on which $w^2(x) < 0$. Each pair of complex conjugate branch points will be joined by a simple arc so that all the cuts in this system intersect the projection of the same real oval. As a model of our curve we take two copies of the Riemann sphere glued crosswise along the cuts we have introduced. The reflection $\bar{J}$ acts as complex conjugation on each sheet. In Fig. 2 we draw the system of cuts in bold and use thin lines for the $g$ odd and $g$ even cycles on the surface (a dashed line means that the part of the contour lies on the lower sheet). For instance, odd cycles go on one sheet along the sides of the cuts introduced; the first $k-1$ of the introduced even cycles are real ovals of the curve. We have a lot of freedom in choosing the last $g-k+1$ even and odd cycles. See [2] for greater detail.

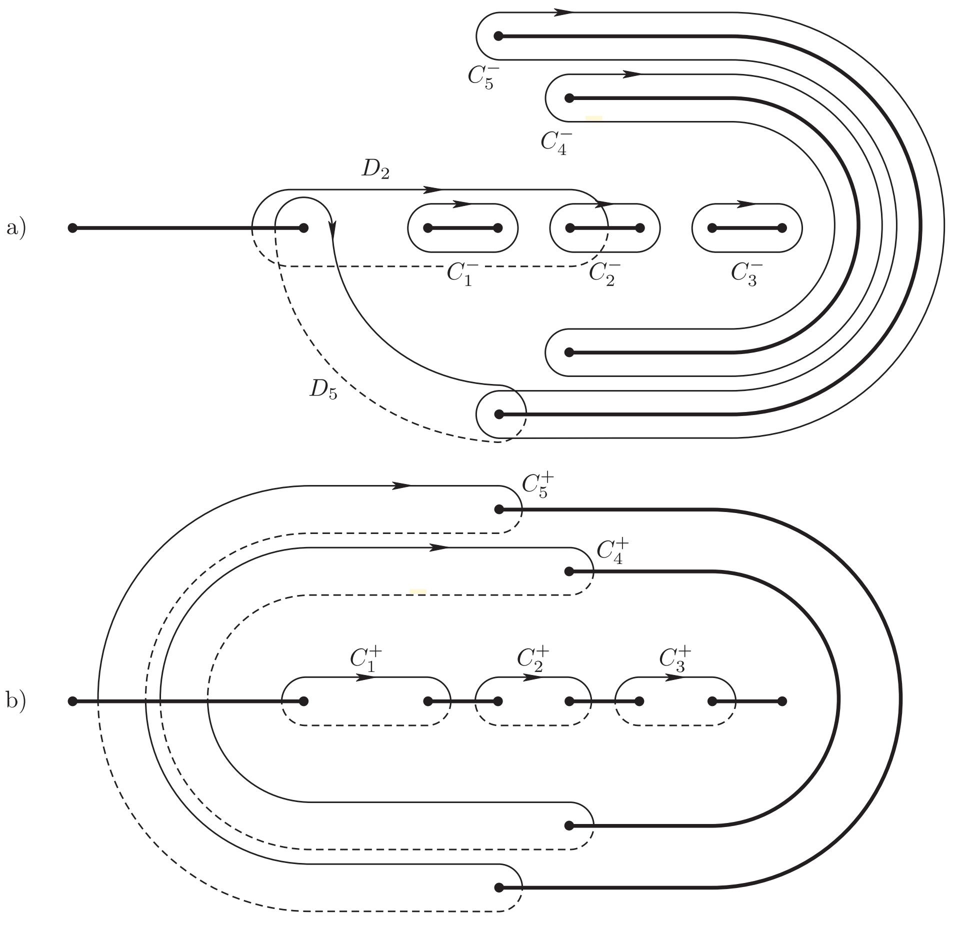


Figure 2. The system of cuts in the plane and the basis in the lattice a) $H_1^-(M,\mathbb{Z})$; b) $H_1^+(M,\mathbb{Z})$.

The only entries of the intersection matrix corresponding to these $2g$ cycles distinct from zero are the following:

$$\begin{aligned} C_s^+ \circ C_s^- &= 1, && s = 1,\dots,k-1, \\ C_s^+ \circ C_{s-1}^- &= -1, && s = 2,\dots,k-1, \\ C_s^+ \circ C_s^- &= 2, && s = k,\dots,g. \end{aligned} \tag{6.1}$$

The intersection matrix is nonsingular, but not unimodal for $k \leqslant g$. So the cycles under consideration form a basis of the real homology, but in general an expansion of an integer cycle with respect to this basis has half-integer coefficients.

**Lemma 4.** *For $k > 0$:*

1) *the cycles $C_1^{\pm}, C_2^{\pm}, \dots, C_g^{\pm}$ form a basis of the lattice $H_1^{\pm}(M,\mathbb{Z})$;*

2) *the cycles $2C_1^{\pm}, 2C_2^{\pm}, \dots, 2C_{k-1}^{\pm}, C_k^{\pm}, \dots, C_g^{\pm}$ form a basis of the lattice $L_M^{\pm}$.*

*Proof.* We shall give the proof for the lattices of odd cycles; for even cycles our arguments will be the same.

1) It is sufficient to produce $g$ linear functionals with values in $\mathbb{Z}$ such that their values at $C_1^-, \dots, C_g^-$ fit into a matrix with determinant $\pm 1$. For these functionals we take the intersection indices with the following integer cycles:

$$
\begin{aligned}
D_1 &:= C_1^+, \\
D_2 &:= C_2^+ + C_1^+, \\
&\dots\dots\dots\dots\dots\dots\dots \\
D_{k-1} &:= C_{k-1}^+ + C_{k-2}^+ + \dots + C_1^+.
\end{aligned}
\tag{6.2}
$$

The remaining cycles $D_j$ are integer solutions of the equation

$$
D_j + \bar{J} D_j = C_j^+, \qquad j = k, \dots, g;
$$

and we have shown one such cycle in Fig. 2, a). For these we have $2D_j \circ C^- = C_j^+ \circ C^-$ for $C^- \in H_1^-$, $j = k, \dots, g$. Now we can recover all the intersection indices from (6.1); it is easy to calculate that the intersection indices $D_s \circ C_j^-$ form the identity matrix.

2) An element of the lattice $L_M^-$ of antisymmetrized integer cycles can be expanded with respect to the basis of the lattice of odd integer cycles introduced above. The coefficients of the expansion are equal to the intersection indices with the cycles $D_j$. Note that $D_1, \dots, D_{k-1}$ are even cycles, so the coefficients at the generators $C_1^-, \dots, C_{k-1}^-$ are even:

$$
(C - \bar{J} C) \circ C^+ = 2C \circ C^+, \qquad C \in H_1(M,\mathbb{Z}), \quad C^+ \in H_1^+(M,\mathbb{Z}).
$$

**6.2. Example.** The torus (4.1) with $\varkappa \in (0,1)$ contains two real ovals. The even and odd homologies are spanned by the cycles $C^+$ and $C^-$ depicted in Fig. 3.

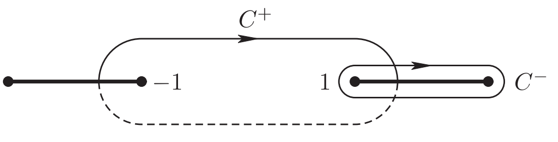


Figure 3. Generators of the lattices of even and odd cycles in the homology space of the torus.

The torus carries a unique holomorphic differential normalized by

$$
\int_{C^+} d\eta := 2 \tag{6.3}
$$

and defining the period

$$\tau := \int_{C^-} d\eta \in i\mathbb{R}. \tag{6.4}$$

We need it in the next section; the differential is real and has the form

$$d\eta = \frac{dr}{2K(\varkappa)s}.$$

**6.3. The case $k = 0$.** The case $k = 0$ never occurs in applications to approximation problems and we only consider it to give a complete picture. A real hyperelliptic curve without real branch points does not necessarily have a real oval (one such example is $w^2 + x^4 + 1 = 0$). Our curve (3.2) has one real oval for even $g$ and two real ovals for odd $g$. In these cases there is no longer any symmetry between even and odd cycles.

As before, assume that complex conjugate branch points are joined by disjoint cuts invariant under reflections in the real axis. Going along the sides of these cuts (Fig. 4) we obtain $g+1$ odd cycles with zero sum. Leaving out one of these cycles, we denote the rest by $C_1^-, C_2^-, \dots, C_g^-$ (see Fig. 4, a)).

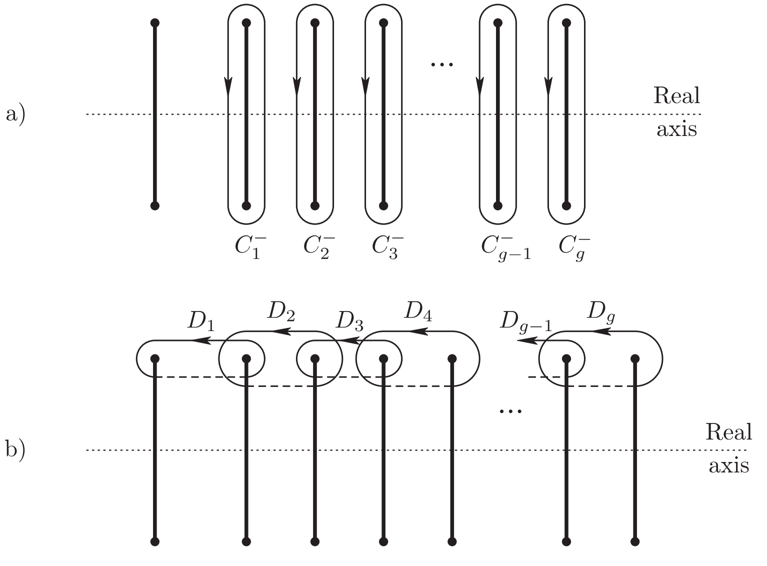


Figure 4. A real hyperelliptic curve without real branch points: a) a basis for odd cycles; b) auxiliary cycles defining a basis of even cycles.

Now we consider the even cycles: $C_j^+ := D_j + \bar{J}D_j$, $j = 1, \dots, 2[g/2]$, where the integer cycles $D_j$ are as represented in Fig. 4, b). For odd $g$ the remaining cycle

$$C_g^+ := D_1 + D_3 + D_5 + \cdots + D_g$$

is homologous to a real oval (either of the two) on the curve. For even $g$ the real oval is homologous to zero.

**Lemma 5.** *For $k=0$:*

1) *the cycles $C_1^\pm, C_2^\pm, \dots, C_g^\pm$ form a basis in the lattice $H_1^\pm(M,\mathbb{Z})$;*

2) *for even $g$ the lattice $L_M^\pm$ coincides with $H_1^\pm(M,\mathbb{Z})$;*

3) *for odd $g$ the lattice $L_M^-$ is formed by integer linear combinations, the sum of whose coefficients is even, of the odd basis cycles introduced above; the cycles $C_1^+$, $C_2^+, \dots, C_{g-1}^+$, $2C_g^\pm$ form a basis of $L_M^+$.*

The proof of this lemma is based on producing integer cycles whose intersection indices with the basis of the corresponding lattice give rise to the identity intersection matrix. We omit it for reasons of space.

## § 7. The main theorem

**Theorem 2.** *A real hyperelliptic curve $M(\mathsf{E})$ is the image of a real rational function $R(x)$ under the Chebyshev correspondence if and only if there is a holomorphic differential $d\zeta$ on the curve with periods satisfying*

$$\int_C d\zeta \in 4\mathbb{Z}, \qquad C \in L_M^+ := (I + \bar{J}_M) H_1(M,\mathbb{Z}), \tag{7.1}$$

$$\int_C d\zeta \in 2\tau\mathbb{Z}, \qquad C \in \begin{cases} L_M^- := (I - \bar{J}_M) H_1(M,\mathbb{Z}), & k>0, \\ H_1^-(M,\mathbb{Z}), & k=0, \end{cases} \tag{7.2}$$

*where the modulus $\tau$ is defined in (6.4).*

*If the inclusions (7.1) and (7.2) hold, the rational function can be recovered by the formula*

$$R(x) = \operatorname{sn}\biggl(2K(\tau)\int_{(e,0)}^{(x,w)} d\zeta + A(e)\biggm|\tau\biggr), \quad A(e) := K(\tau)\begin{cases} \pm 1, & R(e) = \pm 1, \\ \pm 1 + \tau, & R(e) = \pm \dfrac{1}{\varkappa}, \end{cases} \tag{7.3}$$

*in which the result of the calculations is independent of the path of integration on $M$, which of the two possible values of $w(x)$ is chosen and the branch point $e$ taken as the initial point of integration.*

*Remark* 2. 1) Changing the phase term $A(e)$ in the recovery formula (7.3) gives rise to the action of the Klein Vierergruppe on the set of four rational functions $\pm R(x)$, $\pm 1/(\varkappa R(x))$, which generate the same Riemann surface $M(\mathsf{E})$.

2) One curve $M(\mathsf{E})$ can carry several forms $d\zeta$ satisfying (7.1) and (7.2).

3) One of inclusions (7.1) or (7.2) can be regarded as a normalization (definition) of the holomorphic differential; then the other inclusions impose $g$ real constraints on the $2g-1$ real moduli of the Riemann surface $M(\mathsf{E})$.

*Proof of Theorem* 2. 1. Assume that a real rational function generates a Riemann surface. We know that this Riemann surface covers a torus so that the covering intertwines the actions of the hyperelliptic involutions $J_M$ on the curve and $J_T$ on the torus. It is easy to see (for example. from (4.2)) that the covering map also intertwines the reflections $\bar{J}_M$ and $\bar{J}_T$ on the curve and the torus, respectively.

The covering $\widetilde{R}$ lifts the differential $d\eta$ defined in (6.3) from the torus to the surface $M$:

$$d\zeta := \widetilde{R}^* \, d\eta. \tag{7.4}$$

For each integer cycle $C$ on the surface we have

$$\int_{C \pm \bar{J}_M C} d\zeta = \int_{\widetilde{R}C \pm \widetilde{R}\bar{J}_M C} d\eta = \int_{(I \pm \bar{J}_T)\widetilde{R}C} d\eta, \qquad (I \pm \bar{J}_T)\widetilde{R}C \in 2\mathbb{Z}C^{\pm},$$

which yields the inclusions (7.1), and for $k > 0$ also (7.2). For $k = 0$ we must prove the inclusions (7.2) for all odd integer cycles on the surface, not only for antisymmetrized ones.

Recall that, if $k = 0$, we can take the elements of a basis of odd cycles in the form $C = B - JB$, where $B = -\bar{J}_M B$ is a symmetric arc (open chain) joining a pair of branch points. At the end-points of $B$ the real function $R(x)$ takes the same value from the set $\mathsf{Q}$, so $\widetilde{R}B$ is a closed chain on the torus. Correspondingly, $\widetilde{R}C = 2\widetilde{R}B \in 2H_1(T, \mathbb{Z})$ since the hyperelliptic involution acts on the homology by a sign change. Thus we have established (7.2) for $k = 0$ too.

Finally we obtain the recovery formula (7.3) for the rational function. We lift the action of $\widetilde{R}$ to the universal cover of the torus:

$$\int_{(e,0)}^{(x,w)} d\zeta = \int_{\widetilde{R}(e,0)}^{\widetilde{R}(x,w)} d\eta = \eta(\widetilde{R}(x,w)) - \eta(\widetilde{R}(e,0)), \tag{7.5}$$

where $\eta(r,s) := \int_{(0,1)}^{(r,s)} d\eta$ is the (isomorphic) map of the universal cover of the torus onto the complex plane. The function $r(\eta)$ is periodic, with period lattice

$$L = L(\eta) := 2\mathbb{Z} + \tau\mathbb{Z},$$

has simple zeros at $\eta = 0, 1$ and simple poles at $\eta = \tau/2, 1 + \tau/2$, and is normalized by $r(1/2) = 1$. These data suffice for its recovery: $r(\eta) = \operatorname{sn}(2K(\tau)\eta|\tau)$, where $K(\tau)$ is the complete elliptic integral with modulus $\varkappa(\tau)$.

Since the diagram (4.3) is commutative, it follows that

$$R(x) = p_T \circ \widetilde{R}(x,w) = r(\eta(\widetilde{R}(x,w))) = \operatorname{sn}\biggl(2K(\tau)\biggl(\int_{(e,0)}^{(x,w)} d\zeta + \eta(\widetilde{R}(e,0))\biggr)\bigg|\tau\biggr).$$

The phase shift in the last formula is determined from the image $R(e)$ of the distinguished branch point $e$ of the surface:

$$R(e) \in \mathsf{Q}, \qquad \eta(\pm 1, 0) = \pm\frac{1}{2} \pmod{L}, \qquad \eta\biggl(\pm\frac{1}{\varkappa}, 0\biggr) = \pm\frac{1}{2} + \frac{\tau}{2} \pmod{L}.$$

2. Conversely, assume that on the curve $M(\mathsf{E})$ there exists a holomorphic differential with periods satisfying inclusions (7.1) and (7.2). We claim that the right-hand side of (7.3) defines a single-valued function on the sphere. Decomposing an arbitrary integer cycle $C$ on the surface $M(\mathsf{E})$ into a sum of (half-integer) even and odd components, from the embeddings (7.1) and (7.2) we obtain

$$\int_C d\zeta \in L := 2\mathbb{Z} + \tau\mathbb{Z}.$$

Thus the argument of the elliptic sine in (7.3) is defined up to elements of the lattice $4K\mathbb{Z}+2K'\mathbb{Z}$. Therefore, the right-hand side of the formula does not depend on the path of integration joining the branch point $(e,0)$ and the lift $(x,w)$ of the point $x$ to the Riemann surface. The particular lift (that is, picking one of the two possible values of $w(x)$, which are distinct by a factor of $-1$) is also unessential: $\operatorname{sn}(A(e)+u)$ is an even function of $u$, and under holomorphic involution the holomorphic differential $d\zeta$ gets multiplied by $-1$.

Singularities of the function on the right-hand side of (7.3) are at worst powerlike, so it is a rational function of $x$. Now we will show that this function is real. We see from (7.1) that the differential $d\zeta$ has real periods over all the even cycles, so it is itself real, that is, it satisfies the equality $\bar{J}_M\, d\zeta = \overline{d\zeta}$. We have the chain of equalities

$$\begin{aligned}\overline{R(\bar{x})} &= \operatorname{sn}\biggl(2K(\tau)\int_{(\bar{e},0)}^{(x,w)} d\zeta + \overline{A(e)}\biggm|\tau\biggr)\\ &= \operatorname{sn}\biggl(2K(\tau)\int_{(e,0)}^{(x,w)} d\zeta + A(e) + 2K(\tau)\int_{(\bar{e},0)}^{(e,0)} d\zeta\biggm|\tau\biggr).\end{aligned}$$

If $e=\bar{e}$, then there is no additional integral between branch points in the last equality. Otherwise this term is half the integral over some generator of the lattice $L_M^-$ for $k>0$ or of $H_1^-(M,\mathbb{Z})$ for $k=0$; hence

$$\int_{(\bar{e},0)}^{(e,0)} d\zeta \in \tau\mathbb{Z}, \qquad \overline{R(\bar{x})} = R(x).$$

We claim that the curve associated with the resulting rational function $R(x)$ by the Chebyshev correspondence coincides with the original curve $M(\mathsf{E})$. We will find the full inverse images $R^{-1}\mathsf{Q}$ of branch points on the torus $T$. The elliptic sine function takes values in $\mathsf{Q}$ if and only if its argument lies in the translated half-period lattice $(1+L)K(\tau)$. In other words all the points $x$ in the inverse image are determined from the inclusion

$$2\int_{(e,0)}^{(x,w)} d\zeta \in L.$$

All the branch points $x=e_*$ of the original curve lie in $R^{-1}\mathsf{Q}$ because

$$2\int_{(e,0)}^{(e_*,0)} d\zeta = \int_C d\zeta \in L, \qquad C\in H_1(M,\mathbb{Z}).$$

Moreover, the branching indices of $R(x)$ at all the points $x=e_*$ are odd, and at all the other points in $R^{-1}\mathsf{Q}$ they are even: this follows from the expression of $R(x)$ in local variables and the fact that $\operatorname{sn}(W+u)$ is an even function of $u$ for $W\in\operatorname{sn}^{-1}\mathsf{Q}$.

The proof of Theorem 2 is complete.

*Calculating the degree of the rational function.* By contrast with the Chebyshev representation for polynomials, the degree of the rational function is not used explicitly in the recovery formula (7.3). We can find it if we know the map in the homology induced by the covering map from the Riemann surface $M$ onto the torus.

We already know that the covering map intertwines the reflections on the surface and the torus. Hence the even (odd) cycles on the surface are taken to even (odd, respectively) cycles on the torus:

$$\widetilde{R}D_j^+ := m_j^+C^+, \qquad \widetilde{R}C_j^- := m_j^-C^-, \qquad j=1,\dots,g, \quad m_j^\pm \in \begin{cases} \mathbb{Z}, & j<k, \\ 2\mathbb{Z}, & j \geqslant k. \end{cases} \tag{7.6}$$

Here $D_j^+ := C_1^+ + C_2^+ + \dots + C_j^+$ for $j=1,\dots,k-1$ and $D_j^+ := C_j^+$ for $j=k,\dots,g$ is a new basis in the lattice of even cycles on $M$, and its intersection matrix with the basis of odd cycles $C_s^-$ introduced above is diagonal.

**Lemma 6.** *For $k>0$*

$$\deg R = \sum_{j<k} m_j^+ m_j^- + \frac{1}{2}\sum_{j\geqslant k} m_j^+ m_j^-. \tag{7.7}$$

*Proof.* We consider the area 2-form $i\,d\eta\wedge\overline{d\eta}$ on the torus and lift it by the covering map $\widetilde{R}$ to the Riemann surface $M$, where it takes the form $i\,d\zeta\wedge\overline{d\zeta}$. The degree of the cover is equal to the ratio of the areas of the surface and the torus. The area of the surface is related by Riemann's formula (5.3) to the periods along a basis of 1-cycles. Calculating the areas of the torus and the surface we arrive at the required result:

$$\operatorname{vol}(T) = i\int_T d\eta\wedge\overline{d\eta} = -i\biggl(\int_{C^-} d\eta\,\overline{\int_{C^+} d\eta} - \int_{C^+} d\eta\,\overline{\int_{C^-} d\eta}\biggr) = 4\operatorname{Im}\tau > 0,$$

$$\begin{aligned}\operatorname{vol}(M) &= i\int_M d\zeta\wedge\overline{d\zeta} = -i\biggl(\sum_{j<k}\biggl(\int_{C_j^-} d\zeta\,\overline{\int_{D_j^+} d\zeta} - \int_{D_j^+} d\zeta\,\overline{\int_{C_j^-} d\zeta}\biggr)\biggr) \\ &\qquad - i\biggl(\frac{1}{2}\sum_{j\geqslant k}\biggl(\int_{C_j^-} d\zeta\,\overline{\int_{D_j^+} d\zeta} - \int_{D_j^+} d\zeta\,\overline{\int_{C_j^-} d\zeta}\biggr)\biggr) \\ &= 4\operatorname{Im}\tau\biggl(\sum_{j<k} m_j^+m_j^- + \frac{1}{2}\sum_{j\geqslant k} m_j^+m_j^-\biggr).\end{aligned}$$

## § 8. Examples

Now we show how we obtain the classical Zolotarëv fraction (see [7]) in the framework of our construction; we also discuss how to solve other problems of least deviation from pulse functions. The corresponding numerical calculations were performed by D. V. Yarmolich.

**8.1. The Zolotarëv fraction.** We fix a positive integer $n$ and a modulus $\tau\in i\mathbb{R}^+$. In the complex $u$-plane we consider the rectangle

$$\{|\operatorname{Re} u|<1;\ 0<\operatorname{Im} u<n|\tau|\}$$

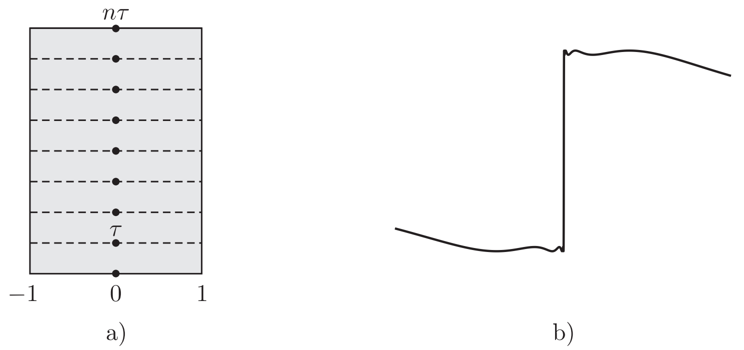


Figure 5. a) A large rectangle formed by smaller ones; b) the graph of the Zolotarëv fraction of degree $n = 8$.

formed by $n$ rectangles of size $1 \times |\tau|$, as in Fig. 5. We denote conformal maps onto the upper half-plane of the larger and smaller rectangles by $x(u)$ and $R(u)$, respectively. After a normalization fixing the three points $u = -1, 0, 1$ these maps become

$$x(u) = \operatorname{sn}(K(n\tau)u|n\tau), \qquad R(u) = \operatorname{sn}(K(\tau)u|\tau), \tag{8.1}$$

but their concrete form is of no importance to us now. The reflection principle shows that the function $R(u)$ mapping the smaller rectangle

$$\{|\operatorname{Re} u| < 1;\ 0 < \operatorname{Im} u < |\tau|\}$$

onto the upper half-plane can be also defined in the larger rectangle and is real on its boundary. From this it is easy to see that $R$ is a (single-valued) meromorphic function of $x$ on the whole of the Riemann sphere, that is, it is a rational function. It is called the *Zolotarëv fraction*; it has simple zeros for $x$ corresponding to $u = 0, \pm 2\tau, \pm 4\tau, \dots$ and simple poles at the points corresponding to $u = \pm\tau, \pm 3\tau, \dots$. All critical points of the rational function $R(u(x))$ lie on the real axis and correspond to

$$u = \pm 1 \pm \tau,\ \pm 1 \pm 2\tau,\ \dots,\ \pm 1 \pm (n-1)\tau;$$

there are precisely $2n - 2$ of them. The critical values belong to the set of four points $\pm 1$, $\pm 1/\varkappa(\tau)$, so the Zolotarëv fraction has a Chebyshev representation with $g = 1$.

Indeed, the torus $\mathbb{C}/(4\mathbb{Z} + 2n\tau\mathbb{Z})$ glued together from four copies of the larger rectangle in Fig. 5, covers the torus $\mathbb{C}/(4\mathbb{Z} + 2\tau\mathbb{Z})$ (glued together from 4 copies of one of the smaller rectangles in Fig. 5) without ramification. In our variables $d\zeta = d\eta = du/2$, and formula (7.3) becomes the parametric representation (8.1).

**8.2. A cut rectangle.** Deforming the construction of the Zolotarëv fraction we can govern the critical values explicitly. This is similar to the way that, using the Akhiezer 'comb', a classical Chebyshev polynomial (see [8]) can be transformed into a Chebyshev polynomial on several intervals. We draw $g-1$ horizontal cuts starting at the boundary of the larger rectangle at heights which are multiples of $\tau$, as in Fig. 6. Let $R(u)$ be the same function as in (8.1) and let $x(u)$ map the cut rectangle

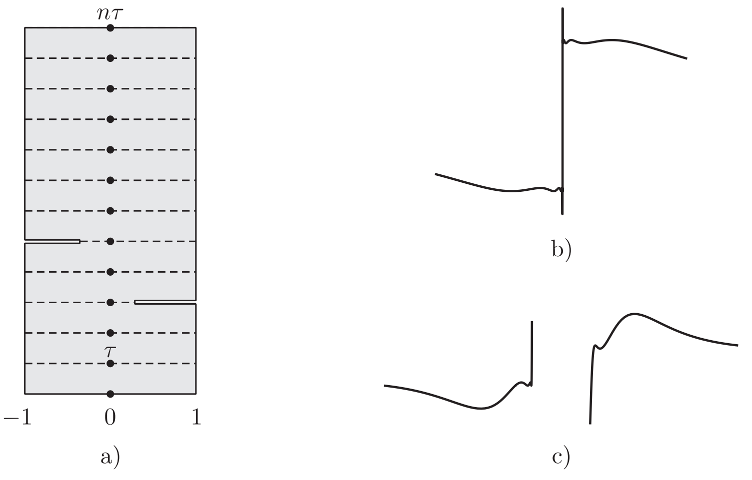


Figure 6. a) The larger rectangle with two cuts; b) the graph of the deformed Zolotarëv fraction of degree $n = 12$; c) the fine structure of the graph to the left and right of the front (rescaling the $x$-axis).

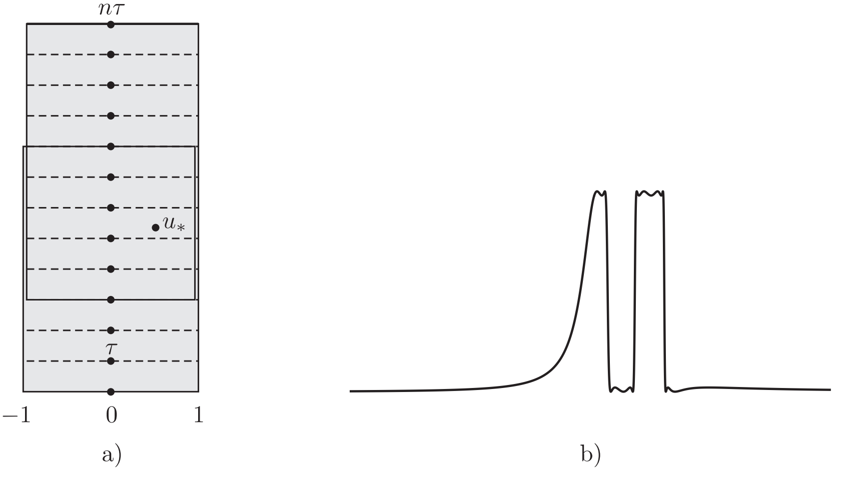


Figure 7. a) The two-sheeted octagonal rectangle; b) the graph of the rational function of degree $n = 12$.

conformally onto the upper half-plane. As in § 8.1, we can readily show that the pair of functions $R(u)$, $x(u)$ defines a rational function on the Riemann sphere parametrically. All the critical points of this function are real and the values taken at them lie in the set $\pm 1$, $\pm 1/\varkappa(\tau)$, apart from the $g-1$ points $x$ corresponding to the end-points of the cuts in the $u$-plane.

This rational function can also be described by means of the Chebyshev construction. The inverse function of $x(u)$ can be expressed by the Schwarz-Christoffel integral, which now is a holomorphic Abelian integral on a hyperelliptic curve of genus $g$. All the branch points of the curve are real; they are $x$-images of the vertices of the cut rectangle which have right angles. We see that the rational function $R(u(x))$ has a representation in the form (7.3).

**8.3. A many-sheeted rectangle.** Now we present an example which has a direct bearing on uniform rational approximation of pulse functions. Consider a two-sheeted rectangular octagon with interior branch point $u_*$, as in Fig. 7. All of its four horizontal sides lie at heights which are multiples of $\tau$. The pair of functions $R(u)$, $x(u)$, where $R(u)$ is as before and $x(u)$ maps the two-sheeted octagon conformally onto the upper half-plane, defines a real rational function $R(u(x))$ parametrically. This function has two complex conjugate critical points $x(u_*)$ and $\overline{x(u_*)}$; its other critical points are real and the values at them belong to the four-element set $\pm 1$, $\pm 1/\varkappa(\tau)$. The rational function obtained is 3-extremal, so it has a representation (7.3) with Abelian integral on a genus 3 curve. This Abelian integral is the Schwarz-Christoffel integral mapping the upper half-plane onto the two-sheeted octagonal rectangle.

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