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MATHEMATICS

Antisymmetric Solutions of Poincaré–Steklov Integral Equations

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A BRIEF STATEMENT OF THE RESULT

Consider the following singular integral Poincaré–Steklov equation (PS-3) with spectral parameter λ:

$$\lambda \text{v.p.}\int_I \frac{u(t)}{t-x}dt - \text{v.p.}\int_I \frac{u(t)dR(t)}{R(t)-R(x)} = \text{const}, \qquad (1)$$

$$x \in I := (-1, 1),$$

where $u(t)$ is the unknown function and the constant does not depend on x. The functional parameter $R(t)$ of the equation is a smooth nondegenerate change of variable on the integration interval:

$$\frac{d}{dt}R(t) \neq 0, \quad t \in [-1, 1]. \qquad (2)$$

By using methods of complex geometry and combinatorics, we obtain constructive visual representations for all so-called antisymmetric solutions [4] of Eq. (1) in which $R(t)$ is a real-valued rational function of degree 3 with separated real critical values different from the endpoints of the integration interval. From the equation, we explicitly construct a Riemann surface with boundary, called a pair of pants, whose conformal class depends on three real numbers. For the number λ and some auxiliary parameters (two reals and several integers), we explicitly construct another pair of pants. It turns out that λ is an eigenvalue of the integral equation (1) with antisymmetric eigenfunction $u(t)$ if and only if the former pair of pants is conformally equivalent to the latter. Solving the spectral problem for the integral equation essentially reduces to solving three transcendental equations with respect to three numbers, the moduli of the pants. Such a representation of solutions cannot be called explicit in the classical sense, but it reveals their interesting properties. In [4], it was used to describe the formation mechanism of a discrete spectrum, obtain sharp boundaries for the spectrum location, and count the zeros of antisymmetric eigenfunctions.

THE ORIGIN OF THE PROBLEM

Boundary value problems for the elliptic equation with a spectral parameter in the boundary conditions were first considered by Poincaré (1895) and Steklov (1901). At present, they have become a popular method for studying dynamics of two-phase fluids, composites, diffraction, etc. One of such boundary value problems arises in substantiating and optimizing the domain decomposition method.

Given a plane domain Ω separated into two simply connected subdomains Ω_1 and Ω_2 by a simple smooth curve Γ, it is required to find a value of the spectral parameter λ at which on each of the subdomains Ω_s (s = 1, 2), there exists a nonzero harmonic function U_s vanishing on the outer boundary $\partial\Omega\backslash\Gamma$. The values of the functions on the interface Γ between the subdomains must coincide ($U_1 = U_2$), and their normal derivatives must differ by a factor of $-\lambda$: $-\lambda \quad = \quad$.

The eigenvalues λ and the traces of the eigenfunctions U_s on the interface Γ are the critical values and the critical points, respectively, of the functional (called the generalized Rayleigh relation)

$$= \frac{|\quad|}{\int_{\Omega_1} |\nabla U_1|^2 d\Omega_1}, \quad U \in H_{oo}^{1/2}(\Gamma),$$

where U_s is the harmonic extension of U from Γ to the subdomain Ω_s with zero Dirichlet data on $\partial\Omega_s\backslash\Gamma$ for s =

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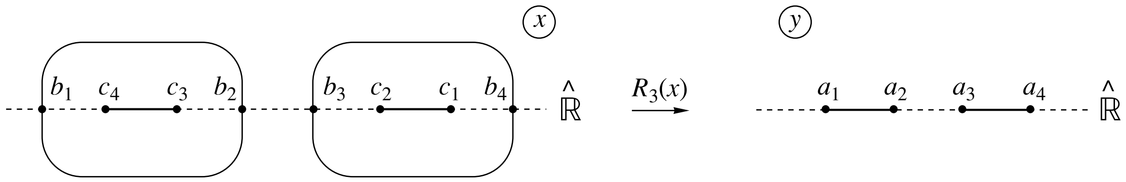

Fig. 1. The topology of the covering R_3 with real branching points.

CLASSIFICATION OF THE PS-3 INTEGRAL EQUATIONS

A third-degree rational function $R(x) = R_3(x)$ determines a triple covering of the Riemann sphere by another sphere, which is generically branched at four points a_s, where $s = 1, 2, 3, 4$. This means that the preimage of a_s consists of a critical point b_s and a regular point c_s. We assume that the four branching points a_s are different reals not equal to ± 1. We enumerate them according to the natural cyclic order on $\hat{\mathbb{R}}$.

The full preimage $R_3^{-1}(\hat{\mathbb{R}})$ consists of the extended real line and two pairs of complex conjugate curves intersecting the real line in the critical points b_1, b_2, b_3, and b_4 as shown in Fig. 1, left. The complement to this set has six components, which are one-to-one mapped onto the upper and lower half-planes.

The functional parameter $R_3(x)$ of the integral equation is a nondegenerate change of variables on the interval $[-1, 1]$. This means, in particular, that none of the critical points b_s belongs to this interval, so that there are two possible cases[1]:

type $\mathcal{A}$: $[-1, 1] \subset [b_2, b_3]$;

type $\mathcal{B}$: $[-1, 1] \subset [b_3, b_4]$.

The remaining cases (e.g., where $[-1, 1]$ is contained in $[b_1, b_2]$ or $[b_4, b_1]$) are eliminated by suitably renumbering the branching points a_s, which thereby acquire unique numbers. In type $\mathcal{B}$, we distinguish between the following subcases:

type $\mathcal{B}$ 1 : $[-1, 1] \cap [c_2, c_1] = \emptyset$,

type $\mathcal{B}$ 21 : $[-1, 1] \subset [c_2, c_1]$,

type $\mathcal{B}$ 22 : $[-1, 1] \supset [c_2, c_1]$,

type $\mathcal{B}$ 23 : $[-1, 1] \subset ([b_3, c_2] \cup [c_1, b_4])$.

[1] A pair of points on the circle breaks the circle into two segments. Our choice of a segment is determined by the natural requirements $b_1, b_2 \notin [b_3, b_4]$; $b_1, b_4 \notin [b_2, b_3]$; $b_3, b_4 \notin [b_1, b_2]$; and so on.

THE PAIR OF PANTS ASSOCIATED TO THE INTEGRAL EQUATION

The functional parameter $R_3(x)$ of the integral equation (1) determines a sphere with three slits (a pair of pants):

$$(\mathcal{P}3) := \mathrm{Cl}\,(\hat{\mathbb{C}} \setminus \{([-1,1]\,[a_1, a_2]) \triangle [a_3, a_4]\}, \quad (3)$$

where $\triangle$ denotes symmetric difference (the union of two sets minus their intersection), and the closure (Cl) of the sphere with cuts is understood (here and in what follows) in the sense of the intrinsic spherical metric, where each cut is supplemented by a pair of coasts. The boundary ovals of the pants are colored as follows: $[-1, 1] \setminus [a_1, a_2]$ is red, $[a_1, a_2] \setminus [-1, 1]$ is blue, and $[a_3, a_4]$ is green. The functional parameter can be reconstructed from the conformal class of the associated pants up to linear-fractional transformations, which do not affect the spectrum and the eigenfunctions of the integral equation.

KLEIN MEMBRANES

In this section, we define pants $\mathcal{P}\mathfrak{Q}(\lambda, h_1, h_2 | m_1, \ldots)$ (see Fig. 2) of several cuts $\mathfrak{Q}$ depending on the spectral parameter λ, two real numbers h_1 and h_2, and one or two integers $m_1, \ldots$. This pair of pants is a many-sheeted surface (Überlagerungsfläche); it is constructed by a certain surgery from annuli. The boundary components

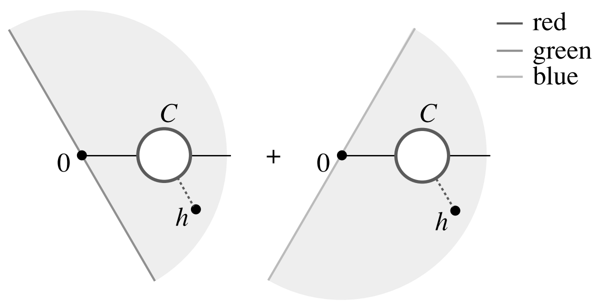

Fig. 2. The pants $\mathcal{P}\mathcal{A}_1(\lambda, h_1, h_2 | m_1, m_2)$ are sewn from two (many-sheeted) annuli.

Table 1. Three-parameter families of pants in cases $\mathcal{A}$ and $\mathcal{B}1$

Cut of pants	Definition	Constraints on the parameters h_1, h_2, m_1, and m_2
$\mathcal{PA}_1(\lambda, h_1, h_2\|m_1, m_2)$	$\mathrm{Cl}\{(m_1 \cdot \alpha)\backslash[\mu^{-1}, h]\} + \mathrm{Cl}\{(m_2 \cdot \alpha)\ [\mu^{-1}, h]\}$	$h := h_1 + ih_2 \in \alpha \cap \bar{\alpha}$, $\|h\| \ge 1$; $m_1, m_2 = 1, 2, \ldots$
$\mathcal{PA}_2(\lambda, h_1, h_2\|m_1, m_2)$	$\mathrm{Cl}\{(m_1 \cdot \alpha)\backslash -\varepsilon^2[h_1, h_2]\} + \mathrm{Cl}\{m_2 \cdot \alpha\}$	$0 < h_1 < h_2$, $h_1h_2 \ge 1$; $m_1 = 1, 2, \ldots, m_2 = 0, 1, 2, \ldots$
$\mathcal{PA}_3(\lambda, h_1, h_2\|m_1, m_2)$	$\mathrm{Cl}\{(m_2 \cdot \alpha)\backslash -\varepsilon[h_1, h_2]\} + \mathrm{Cl}\{m_1 \cdot \alpha\}$	$0 < h_1 < h_2$, $h_1h_2 \ge 1$; $m_1 = 0, 1, 2, \ldots, m_2 = 1, 2, 3, \ldots$
$\mathcal{PB}_1(\lambda, h_1, h_2\|m)$	$\mathrm{Cl}\{(m \cdot \alpha)\backslash[h_1, h_2]\}$	$\mu^{-1} + r < h_1 < h_2$; $m = 1, 2, 3, \ldots$

may lie over five disks and are colored with three colors as follows:

$$C := \{p \in \mathbb{C}: |p - \mu^{-1}|^2 = r^2\} \text{ is red,}$$

$$\varepsilon\hat{\mathbb{R}} \text{ and } \chi(\varepsilon\hat{\mathbb{R}}) \text{ are green,}$$

$$\hat{\mathbb{R}} \text{ and } \varepsilon^2\hat{\mathbb{R}} \text{ are blue;}$$

here, $\varepsilon := \exp\dfrac{2\pi i}{3}$, $\mu := \sqrt{\dfrac{3-\lambda}{2\lambda}}$, $r^2 := \mu^{-2} - 1$, and $\chi(p) := \dfrac{p-\mu}{\mu p - 1}$.

TYPES $\mathcal{A}$ AND $\mathcal{B}1$

The sewing material is two open annuli α and $\bar{\alpha}$ depending on the spectral parameter $\lambda \in (1, 2)$. The annulus α is bounded by the circles C and $\varepsilon\hat{\mathbb{R}}$; the complex conjugate annulus $\bar{\alpha}$ is bounded by the circles C and $\varepsilon^2\hat{\mathbb{R}}$. It is easy to show that, when the spectral parameter varies within the given range, the annuli do not intersect. We denote the unramified m-fold coverings of the annuli α and $\bar{\alpha}$ by $m \cdot \alpha$ and $m \cdot \bar{\alpha}$ for $m = 0, 1, 2, \ldots$.

We define pants of cuts $\mathfrak{Q} = \mathcal{A}_s$ for $s = 1, 2, 3, \mathcal{B}1$ by using Table 1, in which + denotes the surgery described below.

GUIDE FOR SEWING ANNULAR PATCHES

1. $\mathcal{PA}_1$ **($\lambda, h_1, h_2|m_1, m_2$).** Take two (many-sheeted) annuli $m_1 \cdot \alpha$ and $m_2 \cdot \bar{\alpha}$. Cut the upper sheet of each annulus along the same segment (shown by the red dashed line in Fig. 2) starting at the point $h := h_1 + ih_2$ and ending on the circle C. Now, paste each coast of one cut to the opposite coast of the other. The resulting surface is the required pair of pants.

2. $\mathcal{PA}_2$ **($\lambda, h_1, h_2|m_1, m_2$).** The base pair of pants $\mathcal{PA}_2(\lambda, h_1, h_2|m_1, 0)$ is obtained by removing the segment $-\varepsilon^2[h_1, h_2]$ from the upper sheet of the annulus $m_1 \cdot \alpha$. We cut these pants along a segment joining C to the slit (shown by the blue dashed line in Fig. 3). Now, we cut the upper sheet of $m_2 \cdot \bar{\alpha}$ along this segment and crisscross glue together the coast of the resulting cuts. The result is the required pants.

3. $\mathcal{PA}_3$ **($\lambda, h_1, h_2|m_1, m_2$).** The base pair of pants $\mathcal{PA}_3(\lambda, h_1, h_2|0, m_2)$ is obtained by removing the segment $-\varepsilon[h_1, h_2]$ from the upper sheet of the annulus $m_2 \cdot \bar{\alpha}$. As in the preceding case, the required pants are constructed by sewing the many-sheeted annulus $m_1 \cdot \alpha$ to the base pants.

4. $\mathcal{PB}$**1($\lambda, h_1, h_2|m$).** This pair of pant is obtained by deleting the segment $[h_1, h_2]$ from the upper sheet of the annulus $m \cdot \alpha$.

In the limit case, where the branching point $h_1 + ih_2$ tends to $\varepsilon^{\pm 1}\mathbb{R}$, the pants of cut $\mathcal{PA}_1$ coincide with the limit case of the pants $\mathcal{PA}_2$ or $\mathcal{PA}_3$ with $h_1 = h_2 > 0$. The corresponding (unstable) two-parameter families of pants $\mathcal{PA}_{12}$ and $\mathcal{PA}_{13}$ are defined in Table 2.

TYPES $\mathcal{PB}21$, $\mathcal{PB}22$, AND $\mathcal{PB}23$

The two circles $\varepsilon\hat{\mathbb{R}}$ and $\chi(\varepsilon\hat{\mathbb{R}})$ do not intersect provided that $\lambda \in (1, 3)$, and they bound an open annulus β. We denote the unramified m-fold covering of this annulus by $m \cdot \beta$ for $m = 1, 2, 3, \ldots$ and represent its points in the form

$$p = \mu^{-1} + \rho\exp(i\phi),$$

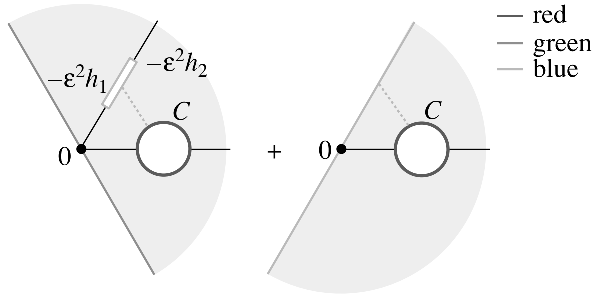

Fig. 3. The pants $\mathcal{PA}_2(\lambda, h_1, h_2|m_1, m_2)$ are obtained by sewing the annulus $m_2 \cdot \bar{\alpha}$ into the base pants.

Table 2. Two-parameter families of pants for the parameter ranges $1 < \lambda < 2$, $h > 0$, and $m_1, m_2 = 1, 2, 3, \ldots$

Cut of pants	Definition
$\mathcal{PA}_{12}(\lambda, h\vert m_1, m_2)$	$\mathcal{PA}_1(\lambda, -\mathrm{Re}(\varepsilon^2 h), -\mathrm{Im}(\varepsilon^2 h)\vert m_1, m_2) = \mathcal{PA}_2(\lambda, h, h\vert m_1, m_2)$
$\mathcal{PA}_{13}(\lambda, h\vert m_1, m_2)$	$\mathcal{PA}_1(\lambda, -\mathrm{Re}(\varepsilon h), -\mathrm{Im}(\varepsilon h)\vert m_1, m_2) = \mathcal{PA}_3(\lambda, h, h\vert m_1, m_2)$

Table 3. Slits in the annulus $m \cdot \beta$

Definitions of slits	The range of h_1 and h_2
$E_1^1(h_1) := \mu^{-1} + r\exp[-h_1, h_1]$ $E_1^2(h_2) := \mu^{-1} + r\exp[-h_2, h_2]\exp(i\pi m)$	$h_1 \geq h_2 > 0$ if m is even, $(\mu^{-1} + r\exp h_1)\cdot(\mu^{-1} - r\exp h_2) \geq 1$ if m is odd
$E_2^1(h_1) := \mu^{-1} + r\exp[-ih_1, ih_1]$ $E_2^2(h_2) := \mu^{-1} + r\exp[-ih_2, ih_2]\exp(i\pi m)$	$h_1 \geq h_2$ if m is even, $\mathrm{Arg}(\exp(ih_1) + \mu r) \geq \mathrm{Arg}(\exp(ih_2) - \mu r)$ if m is odd, $h_1 + h_2 < m\pi$ and $h_2 > 0$ for all m
$E_3^1(h_1) := \mu^{-1} + r\exp[-h_1, h_1]$ $E_3^2(h_2) := \mu^{-1} + r\exp[-ih_2, ih_2]\exp(i\pi m)$	$h_1 > 0$, $m\pi > h_2 > 0$

where $\rho > 0$ and $\phi \in \mathbb{R}\bmod 2\pi m$. The action of the linear-fractional transformation χ on the sphere (that is, successive reflections in the circles C and $\hat{\mathbb{R}}$) is lifted to the involution of the many-sheeted annulus $m \cdot \beta$ defined by

$$\Xi\colon \mu^{-1} + \rho\exp(i\phi) \to \mu^{-1} + \frac{r^2}{\rho}\exp(-i\phi), \tag{4}$$

where $r := \sqrt{\mu^{-2} - 1}$ is the radius of the circle C.

Consider pants of cuts $\mathfrak{Q} = \mathcal{B}21$, $\mathcal{B}22$, and $\mathcal{B}23$, which depend on the real numbers $\lambda \in (1, 3)$, h_1, and h_2 and the positive integer m.

Definition. For $s = 1, 2, 3$, we set $\mathcal{PB}2s(\lambda, h_1, h_2|m) :=$ $\mathrm{Cl}\{(m\cdot\beta)\backslash(E_s^1(h_1) \cup E_s^2(h_2))\}/\Xi$, where E_s^1 and E_s^2 are the slits defined in Table 3. These slits are invariant with respect to the involution Ξ and do not intersect each other and the boundaries of the annulus $m \cdot \beta$.

ANTISYMMETRIC SOLUTIONS

The spectrum of the integral equation is positive [1], because this is the set of critical points of a positive functional (the ratio of two Dirichlet integrals). Without loss of generality, we seek only real eigenfunctions. There are two kinds of them, symmetric and antisymmetric eigenfunctions [4]; they differ in geometric contents. In this paper, we consider only antisymmetric eigenfunctions $u(x)$, whose defining feature is the nonpositivity of the value

$$(\lambda - 1)\sum_{k=1}^{3}\Phi(x_k)^2 - 2\sum_{j<s}^{3}\Phi(x_j)\Phi(x_s), \tag{5}$$

where

$$\Phi(x) := \int_I u(t)(t-x)^{-1}dt + \frac{\int_I u(t)d\ln Q(t) - \mathrm{const}}{\lambda - 3},$$

$$x \in \hat{\mathbb{C}}\backslash[-1, 1],$$

the constant const is the same as in Eq. (1), $Q(t)$ is the denominator of the irreducible representation of $R_3(t)$ as a fraction of two polynomials, and $\{x_1, x_2, x_3\} := := R_3^{-1}(y)$. The quantity (5) does not depend on the choice of $y \in \hat{\mathbb{C}}\backslash[-1, 1]$.

MAIN THEOREM

Theorem 1. *If* $\lambda \neq 1, 3$, *then there is a one-to-one correspondence between the antisymmetric eigenfunctions of the* PS-3 *integral equation of types* $\mathfrak{Q} = \mathcal{A}$, $\mathcal{B}1$, $\mathcal{B}21$, $\mathcal{B}22$, $\mathcal{B}23$ *and the pants*[2] $\mathcal{P}\mathfrak{Q}(\lambda, h_1, h_2|m_1, \ldots)$, *which are conformally equivalent to pants* (3) *related to the functional parameter of the integral equation.*

Let $p(y)$ *be a conformal mapping of the pants* $\mathcal{P}(R_3)$ *to the pants* $\mathcal{P}\mathfrak{Q}(\lambda, h_1, h_2|m_1, \ldots)$ *preserving the colors*

[2] In case $\mathcal{A}$, there are three stable and two unstable cuts of pants $\mathcal{PA}_*(\ldots)$.

of the boundary ovals. Then, the eigenfunction has the representation (*up to proportionality*)

$$u(x) = \begin{cases} \sqrt{\dfrac{(y-y_1)(y-y_2)}{p'(y^+)p'(y^-)}}\dfrac{p(y^+)-p(y^-)}{w(y)}, \\ x \in [-1, 1] \backslash [a_1, a_2], \\ \sqrt{(y-y_1)(y-y_2)}\dfrac{\mathrm{Im}\, p(y^+)}{w(y)|p(y^+)|}, \\ x \in [-1, 1] \cap [a_1, a_2]. \end{cases} \quad (6)$$

Here, $y := R_3(x)$ *and* $y^{\pm} := y \pm i0$. *For the cut* $\mathfrak{Q} = \mathcal{A}_1$, $y_1 = \bar{y}_2$ *is an interior critical point of the function* $p(y)$; *for the other cuts* $\mathfrak{Q}$, *the reals* y_1 *and* y_2 *are boundary critical points of the function* $p(y)$.

The approach of this paper is based on reducing the integral equation to Riemann's monodromy problem [2]. The latter can be reformulated as a problem for branched complex projective structures [3] and solved for antisymmetric eigenfunctions by using Grothendieck's technique of child's drawings. The proof of the main theorem for case $\mathcal{A}$ is given in [4] and for case $\mathcal{B}$, in preprint [5].

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