ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2017, Vol. 298, pp. 60–67. © Pleiades Publishing, Ltd., 2017.
Published in Russian in Trudy Matematicheskogo Instituta imeni V.A. Steklova, 2017, Vol. 298, pp. 67–74.

Closed Formula for the Capacity of Several Aligned Segments

A. B. Bogatyrev[a,b,c] **and O. A. Grigoriev**[a]

Received November 21, 2016

Abstract—We present a universal closed formula in terms of theta functions for the Log-capacity of several segments on a line. The formula for two segments was obtained by N. Achieser (1930); three segments were considered by T. Falliero and A. Sebbar (2001).

DOI: 10.1134/S0081543817060050

For physicists the capacity is a coefficient relating the charge and the voltage of a condenser, or equivalently the energy of an electric field inside the condenser under the unit voltage. For mathematicians the capacity is a certain characteristic of a set which shows how "massive" it is. This concept turned out to be very useful in approximation theory, geometric function theory, partial differential equations, and potential theory, to name a few. For a compact set in the complex plane, its capacity coincides with the Chebyshev constant, the transfinite diameter, and the conformal radius (for simply connected sets), and a simple formula links it to the Robin constant of the set (see [1, 9, 12]).

Approaches to the construction of numerical methods aimed at calculating the capacity are as various as the areas of mathematics where this concept emerges. There are rather general techniques suitable in a generic setting, such as those described in [18, 17, 3]. In the present paper we will discuss the capacity of a union of segments, so let us briefly recall the available results concerning this case. A formula expressing the capacity of a set of the indicated kind as an abelian integral was introduced by Widom in [18]. In [2] an expression for the capacity of a system of segments was found in terms of functions arising from Toda's periodic chain equations. Yet another approach was presented in [14], where the capacity was given in terms of automorphic functions. Finally, in [1, 10], explicit expressions for the capacity of a union of two and three segments, respectively, were found in terms of theta functions on a certain Riemann surface; it is these results that will be generalized below (in [10] this technique is further developed to cover the case of an arbitrary number of segments; however, to use the formulas obtained there, one should make some additional preliminary calculations).

It is convenient to have a few different ways to calculate the same value (capacity in this case) not only because of different settings in which the value may appear, but also because many numerical methods for problems arising in complex analysis are plagued by computational instability; for example, this is the case when one searches for the roots of a polynomial or for the images of points under a conformal mapping (notorious crowding phenomenon). If there exist several techniques, one can choose the most stable of them.

[a] Institute of Numerical Mathematics, Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119333 Russia.
[b] Moscow Institute of Physics and Technology (State University), Institutskii per. 9, Dolgoprudnyi, Moscow oblast, 141701 Russia.
[c] Moscow State University, Moscow, 119991 Russia.
E-mail addresses: ab.bogatyrev@gmail.com (A.B. Bogatyrev), guelpho@mail.ru (O.A. Grigoriev).

Let E be a collection of $g+1$ segments on a real line,

$$E := \bigcup_{j=0}^{g} [e_{2j+1}, e_{2j+2}], \tag{1}$$

with a strictly increasing sequence of endpoints e_j. We know that the capacity of a set is independent of its translations and is homogeneous with respect to dilations, so without loss of generality we set the leftmost point of E to be 0 and the rightmost point to be 1:

$$e_1 = 0, \qquad e_{2g+2} = 1. \tag{2}$$

The capacity $C := \operatorname{Cap}(E)$ is defined in terms of the asymptotics of the Green function $G_E(x)$ of this set at infinity:

$$G_E(x) = \log|x| - \log C + o(1), \qquad x \to \infty. \tag{3}$$

We are going to derive a closed formula for $\operatorname{Cap}(E)$ in terms of Riemann theta functions, so we have to introduce some necessary (but standard) constructions related to Riemann surfaces.

1. RIEMANN SURFACES AND THETA FUNCTIONS

The Riemann surface associated with our problem is the double cover of the sphere ramified over the endpoints of the segments from E. This is a genus g compact surface $\mathcal{X}$ whose affine part is given by the equation

$$w^2 = \prod_{j=1}^{2g+2} (x - e_j). \tag{4}$$

The surface admits the hyperelliptic involution $J(x,w) := (x,-w)$ with $2g+2$ fixed points $P_s = (e_s, 0)$ as well as an anticonformal involution (reflection) $\overline{J}(x,w) := (\overline{x}, \overline{w})$. The fixed points of the reflection $\overline{J}$ make up the *real ovals* of the surface; the fixed points of another anticonformal involution $J\overline{J}$ are known as *coreal ovals*. Coreal ovals cover the set E; real ovals cover the complement $\widehat{\mathbb{R}} \setminus E$; in particular, the two points $\infty_\pm := (+\infty, \pm\infty)$ from the same real oval cover the infinity.

We introduce a symplectic basis in the homology of $\mathcal{X}$ as shown in the figure. The dual basis of holomorphic differentials satisfies the normalization conditions

$$\int_{a_j} du_s := \delta_{js} \tag{5}$$

and generates the period matrix

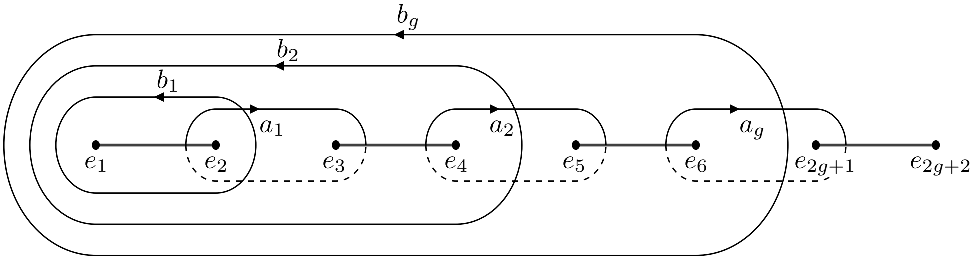
$$\int_{b_j} du_s =: \Pi_{js}. \tag{6}$$

One easily checks that the introduced bases of cycles and differentials behave under the involutions as follows:

$$\overline{J}a_s = a_s, \qquad \overline{J}b_s = -b_s, \qquad Ja_s = -a_s, \qquad Jb_s = -b_s, \tag{7}$$

$$\overline{J}du_s = \overline{du_s}, \qquad Jdu_s = -du_s, \qquad s = 1, \dots, g. \tag{8}$$

The cycles that survive after the reflection $\overline{J}$ are called *even*, and those that change sign are called *odd*. The differentials with the property (8) are usually called *real*, and their periods along even/odd



Symplectic basis in the 1-homology of the curve $\mathcal{X}$.

cycles are real/pure imaginary, respectively (see [4, 6] for more details). In particular, our period matrix Π is pure imaginary. Standard facts are the symmetry of this matrix and positive definiteness of the imaginary part [11].

Given the period matrix, we define the complex g-dimensional torus known as a Jacobian of the complex curve $\mathcal{X}$:

$$\mathrm{Jac}(\mathcal{X}) := \mathbb{C}^g/L(\Pi), \qquad L(\Pi) := \mathbb{Z}^g + \Pi\mathbb{Z}^g. \tag{9}$$

The Abel–Jacobi (AJ) map embeds the curve in its Jacobian:

$$u(P) := \int_{P_1}^{P} du \bmod L(\Pi) \in \mathrm{Jac}(\mathcal{X}), \qquad du := (du_1, du_2, \dots, du_g)^{\mathrm{t}}. \tag{10}$$

It is convenient to represent the points $u \in \mathbb{C}^g$ as the so-called theta characteristics, i.e., pairs of real g-vector columns ϵ, ϵ':

$$u = \frac{1}{2}(\epsilon' + \Pi\epsilon). \tag{11}$$

The points of the Jacobian in this notation correspond to two vectors with real entries modulo 2. The second-order points of the Jacobian are $2 \times g$ matrices with binary entries. In particular, the images of the fixed points P_s of the hyperelliptic involution under the AJ map (10) are as presented in Table 1, where E_s and Π_s are the columns of the identity matrix and the period matrix, respectively. Note that the representation of the theta characteristic as two column vectors written one after another is not universally accepted. Sometimes the transposed matrix is used.

The following absolutely convergent series is known as a theta function:

$$\theta(u;\Pi) := \sum_{m\in\mathbb{Z}^g} \exp\{2\pi i m^{\mathrm{t}} u + \pi i m^{\mathrm{t}}\Pi m\}, \qquad u \in \mathbb{C}^g, \quad \Pi = \Pi^{\mathrm{t}} \in \mathbb{C}^{g\times g}, \quad \mathrm{Im}\,\Pi > 0. \tag{12}$$

Often it is convenient to consider a theta function with characteristics, which is a slight modification of the latter:

$$\begin{aligned}\theta[2\epsilon, 2\epsilon'](u,\Pi) &:= \sum_{m\in\mathbb{Z}^g} \exp\big\{2\pi i(m+\epsilon)^{\mathrm{t}}(u+\epsilon') + \pi i(m+\epsilon)^{\mathrm{t}}\Pi(m+\epsilon)\big\}\\ &= \exp\big\{i\pi\epsilon^{\mathrm{t}}\Pi\epsilon + 2i\pi\epsilon^{\mathrm{t}}(u+\epsilon')\big\}\,\theta(u+\Pi\epsilon+\epsilon';\Pi), \qquad \epsilon,\epsilon' \in \mathbb{R}^g. \end{aligned}\tag{13}$$

The matrix argument Π of the theta function may be omitted if this does not lead to confusion. The vector argument u may also be omitted, which means that we set $u = 0$; the value of the theta function is called the theta constant in this case (note that the dependence on the period matrix is preserved).

Table 1. Images of the fixed points P_s of the hyperelliptic involution under the AJ map (10)

P_s	$u(P_s) \bmod L(\Pi)$	$[\epsilon, \epsilon']^{\mathrm{t}}$	P_s	$u(P_s) \bmod L(\Pi)$	$[\epsilon, \epsilon']^{\mathrm{t}}$
P_1	0	$\begin{bmatrix} 0\,0\ldots0 \\ 0\,0\ldots0 \end{bmatrix}$	P_6	$(\Pi_3 + E_1 + E_2)/2$	$\begin{bmatrix} 0\,0\,1\,0\ldots \\ 1\,1\,0\,0\ldots \end{bmatrix}$
P_2	$\Pi_1/2$	$\begin{bmatrix} 1\,0\,0\ldots \\ 0\,0\,0\ldots \end{bmatrix}$	$\vdots$	$\vdots$	$\vdots$
P_3	$(\Pi_1 + E_1)/2$	$\begin{bmatrix} 1\,0\,0\ldots \\ 1\,0\,0\ldots \end{bmatrix}$	P_{2g}	$(\Pi_g + E_1 + E_2 + \ldots + E_{g-1})/2$	$\begin{bmatrix} 0\ldots0\,1 \\ 1\ldots1\,0 \end{bmatrix}$
P_4	$(\Pi_2 + E_1)/2$	$\begin{bmatrix} 0\,1\,0\ldots \\ 1\,0\,0\ldots \end{bmatrix}$	P_{2g+1}	$(\Pi_g + E_1 + E_2 + \ldots + E_g)/2$	$\begin{bmatrix} 0\ldots0\,1 \\ 1\,1\ldots1 \end{bmatrix}$
P_5	$(\Pi_2 + E_1 + E_2)/2$	$\begin{bmatrix} 0\,1\,0\ldots \\ 1\,1\,0\ldots \end{bmatrix}$	P_{2g+2}	$(E_1 + E_2 + \ldots + E_g)/2$	$\begin{bmatrix} 0\,0\ldots0 \\ 1\,1\ldots1 \end{bmatrix}$

The function (12) has the following easily checked quasi-periodicity properties with respect to the lattice $L(\Pi)$:

$$\theta(u + m' + \Pi m; \Pi) = \exp\{-i\pi m^{\mathrm{t}}\Pi m - 2i\pi m^{\mathrm{t}} u\}\, \theta(u; \Pi), \qquad m, m' \in \mathbb{Z}^g. \tag{14}$$

The theta functions with characteristics have similar transformation rules [15, 16], which can easily be derived from the above formula (14), and we omit them for brevity.

Remark 1. (i) The theta function with integer characteristics $[\epsilon, \epsilon']$ is either even or odd depending on the parity of the inner product $\epsilon^{\mathrm{t}} \cdot \epsilon'$. In particular, all odd theta constants are zeros.

(ii) Adding a matrix with even entries to integer theta characteristics can at worst spoil the sign of the theta function. Hence, the binary arithmetic plays an important role in the calculus of theta functions.

The theta function can be considered as a multivalued function on the Jacobian or as a section of a certain line bundle. The zero set of a theta function—the theta divisor—is well defined on the Jacobian since the exponential factor on the right-hand side of (14) does not vanish. The theta divisor is described by so-called Riemann vanishing theorems [16, 11]. One of the important ingredients in those theorems is a vector of Riemann constants $\mathcal{K}$, which depends on the choice of a homology basis and the initial point of the AJ map. In the above setting the vector of Riemann constants can be found by a straightforward computation [15] or by some combinatorial argument [11] and corresponds to a characteristic

$$\mathcal{K} \sim \begin{bmatrix} \ldots 1\,1\,1\,1\,1 \\ \ldots 1\,0\,1\,0\,1 \end{bmatrix}^{\mathrm{t}}.$$

The zeros of the theta function transferred to the surface by the AJ map are described by the following

Theorem 1 (Riemann). *Let D_g be a degree g positive nonspecial divisor on the curve $\mathcal{X}$. Then the function $\theta(u(P) - \mathcal{K} - u(D_g))$ of the argument $P \in \mathcal{X}$ has exactly g zeros at the points of D_g, counting multiplicities.*

Here and in what follows we use the standard notions of the function theory on Riemann surfaces [11, 16, 15]. A *divisor* is a finite set of points on a surface taken with integer weights: $D = \sum_j m_j P_j$, $m_j \in \mathbb{Z}$, $P_j \in \mathcal{X}$. The *degree of a divisor* is the sum of all weights of the points involved: $\deg D := \sum_j m_j$. A *positive divisor* is a divisor with nonnegative weights only.

The *index of speciality* $i(D)$ of a positive divisor D is the dimension of the space of holomorphic differentials that vanish at the points of D (counting multiplicities). The divisors with index

$i(D) > g - \deg D$ are called *special*, which means that their points are not in general position. Two divisors are said to be *linearly equivalent* if their difference is the set of zeros and poles (the latter should be taken with negative weights) of a meromorphic function. The Riemann–Roch theorem [11] states that the class of a positive divisor contains a unique element if and only if this divisor is not special.

2. GREEN'S FUNCTION

The goal of this section is to give a closed expression for the Green function for a collection E of aligned segments. There exists a unique third-kind abelian differential $d\eta$ with simple poles at two distinguished points $\infty_\pm$ of the curve $\mathcal{X}$ with residues ∓ 1, respectively, and pure imaginary periods. The normalization conditions for this differential imply that it is real, and therefore its a-periods (along even cycles) vanish [6].

Lemma 1. *The Green function for E has the representation*

$$G_E(x) = \left| \mathrm{Re}\left(\int_Q^P d\eta \right) \right|, \tag{15}$$

where the upper limit P covers the argument x of the Green function, $x(P) = x$, and the lower limit Q lies on a coreal oval, $x(Q) \in E$.

Proof. We lift the complement $\widehat{\mathbb{C}} \setminus E$ to the Riemann surface (4) so that the infinity is mapped to the distinguished point ∞_+ of $\mathcal{X}$, and call it the upper sheet of the surface. The other lift will be called the lower sheet. The function equal to $G_E(x)$ on the upper sheet of the surface and to $-G_E(x)$ on the lower sheet is a harmonic function with two log poles at $\infty_\pm$. It is a real part of some third-kind abelian integral that satisfies the same normalization conditions as the integral $\eta = \int d\eta$ introduced above. It remains to check that the integral of $d\eta$ along any coreal oval is pure imaginary. This holds because the distinguished differential is real. □

A similar expression (written in slightly different terms) was used in [18, 2] to calculate the capacity of a system of segments. An explicit formula for the third-kind abelian integrals in terms of theta functions was given by Riemann [11, 16].

Theorem 2.

$$G_E(x) = \left| \log \left| \frac{\theta[\epsilon, \epsilon'](u(P) + u(\infty_+))}{\theta[\epsilon, \epsilon'](u(P) - u(\infty_+))} \right| \right|,$$

where $x(P) = x$ and $[\epsilon, \epsilon']$ is the integer (odd nonsingular) theta characteristic corresponding to the half-period $\mathcal{K} + u(D_{g-1})$, with D_{g-1} a sum of any $g - 1$ different branch points: $D_{g-1} = \sum_{s \in I} P_s$, $I \subset \{1, 2, \dots, 2g+2\}$, $\#I = g - 1$.

Proof. The proof is based on Riemann's theorem on the zeros of theta functions: the divisors $D_g^\pm := D_{g-1} + \infty_\pm$ are nonspecial; therefore, the ratio

$$\frac{\theta(u(P) - \mathcal{K} - u(D_g^-))}{\theta(u(P) - \mathcal{K} - u(D_g^+))}$$

as a multivalued function of $P \in \mathcal{X}$ has a unique zero at $P = \infty_-$ and a unique pole at $P = \infty_+$. The function $\exp\{\int^P d\eta\}$ has the same zero and pole; moreover, both functions acquire the same factors when the argument P goes around the cycles of the surface (this follows from the transformation rules of theta functions (14) and Riemann bilinear relations [11]). Now the Liouville theorem implies that these functions differ by a constant nonzero factor. The usage of theta characteristics allows us to simplify the expression for the abelian integral. □

3. MAIN FORMULA

Up to this moment we have followed (more or less closely) the logic of the paper [10]. Falliero and Sebbar then inserted the expression for the Green function into formula (3) for capacity and used the L'Hôpital rule to resolve the uncertainty $0/0$. This led to the appearance of the coefficients of abelian differentials as well as the derivatives of theta functions. A subsequent struggle with those terms (including a higher genus generalization of Jacobi's formula for the theta derivative) took a lot of efforts and eventually prevented the authors from getting a compact formula for the capacity in the case $g > 2$.

The formula for the capacity contains yet another ingredient along with the Green function: the independent variable x. The representation for the hyperelliptic projection of the surface (4) to the sphere in terms of the variables of the Jacobian is well known [11].

Theorem 3.

$$x(P) = \frac{\theta^2[\epsilon,\epsilon'](u(P))}{\prod_\pm \theta[\epsilon,\epsilon'](u(P)\pm u(\infty_+))}\,\frac{\theta^2[\xi,\xi'](u(\infty_+))}{\theta^2[\xi,\xi']},$$

where $[\epsilon,\epsilon']$ *is again the integer theta characteristic corresponding to the half-period* $\mathcal{K}+u(D_{g-1})$, D_{g-1} *is a positive divisor of any* $g-1$ *different branch points except for the first and last ones, and* $[\xi,\xi'] = [\epsilon,\epsilon'] + [(0\ \ldots\ 0)^{\mathrm{t}}, (1\ \ldots\ 1)^{\mathrm{t}}] \bmod 2$.

Proof. The function $x(P)$ on the surface $\mathcal{X}$ has a double pole at $P = P_1$ and two poles at $P = \infty_\pm$, just as the ratio $\theta^2(u(P)-\mathcal{K}-u(D_{g-1}))/\prod_\pm \theta(u(P)-\mathcal{K}-u(D_{g-1})\pm u(\infty_+))$; this follows from Riemann's theorem on zeros of thetas. The latter expression is single-valued on the surface and therefore differs by a constant factor from the projection $x(P)$. This factor can be found by evaluating both functions at $P = P_{2g+2}$. The resulting expression can be simplified by the usage of theta characteristics. $\square$

Let us gather the previously obtained formulas:

$$\operatorname{Cap}(E) = \lim_{x\to\infty} |x|\exp\{-G_E(x)\} = \lim_{P\to\infty_\pm} |x(P)| \left|\frac{\theta[\epsilon,\epsilon'](u(P)\mp u(\infty_+))}{\theta[\epsilon,\epsilon'](u(P)\pm u(\infty_+))}\right|$$

$$= \left|\frac{\theta[\epsilon,\epsilon'](u(\infty_+))\,\theta[\xi,\xi'](u(\infty_+))}{\theta[\epsilon,\epsilon'](2u(\infty_+))\,\theta[\xi,\xi'](0)}\right|^2, \tag{16}$$

where $[\epsilon,\epsilon']$ is the integer theta characteristic corresponding to the half-period $\mathcal{K}+u(D_{g-1})$, D_{g-1} is a positive divisor of any $g-1$ different branch points except for the first and last ones, and $[\xi,\xi'] = [\epsilon,\epsilon'] + [(0\ \ldots\ 0)^{\mathrm{t}}, (1\ \ldots\ 1)^{\mathrm{t}}] \bmod 2$.

To make this formula computationally effective, one has to calculate the period matrix Π and the image of infinity in the Jacobian $u(\infty_+)$. However, there is a way to avoid the usage of quadrature rules (see, e.g., the technique developed in [5, 13, 7]). The latter approach is particularly useful in the cases close to degenerate ones, when the segments of E tend to vanish or merge.

4. TESTING THE MAIN FORMULA

To verify the above formula (16), we need a good stock of sets E with analytically known capacities. We take the inverse polynomial images of segments as such sets. Indeed, let $E := T^{-1}([-1,1])$, with

$$T(x) = cx^n + (\text{lower degree terms})$$

Table 2. The period matrix of the associated curve $\mathcal{X}$ and the capacity Cap(E) calculated by formula (16) for several sets $E = T_5^{-1}(I)$

I	Period matrix	Capacity
$[-1/6, 1/3]$	$\begin{pmatrix} 0.581304428690 & 0.265097452123 & 0.142465655111 & 0.063806949583 \\ 0.265097452123 & 0.698635810128 & 0.328904401707 & 0.141377608521 \\ 0.142465655111 & 0.328904401707 & 0.723770083801 & 0.265097452123 \\ 0.063806949583 & 0.141377608521 & 0.265097452123 & 0.557258201607 \end{pmatrix}$	0.37892914163
$[-0.5, 0.5]$	$\begin{pmatrix} 0.703235096283 & 0.285878284249 & 0.148276665778 & 0.065947018040 \\ 0.285878284249 & 0.851511762061 & 0.351825302289 & 0.148276665778 \\ 0.148276665778 & 0.351825302289 & 0.851511762061 & 0.285878284249 \\ 0.065947018040 & 0.148276665778 & 0.285878284249 & 0.703235096284 \end{pmatrix}$	0.43527528164
$[-0.2, 0.4]$	$\begin{pmatrix} 0.581799961010 & 0.270529331631 & 0.143046124396 & 0.064421403944 \\ 0.270529331631 & 0.759072205590 & 0.334950735576 & 0.144355718924 \\ 0.143046124397 & 0.334950735576 & 0.724846085406 & 0.270529331631 \\ 0.064421403944 & 0.144355718924 & 0.270529331631 & 0.614716486665 \end{pmatrix}$	0.39300154279
$[-0.1, 0.4]$	$\begin{pmatrix} 0.549252298597 & 0.265164827067 & 0.140966628599 & 0.063811919424 \\ 0.265164827067 & 0.736018990002 & 0.328976746491 & 0.142930161909 \\ 0.140966628599 & 0.328976746491 & 0.690218927196 & 0.265164827067 \\ 0.063811919424 & 0.142930161909 & 0.265164827067 & 0.593088828092 \end{pmatrix}$	0.37892914163
$[-0.5, 0.25]$	$\begin{pmatrix} 0.615250639522 & 0.277253172081 & 0.144977134658 & 0.065127616053 \\ 0.277253172081 & 0.811694974260 & 0.342380788133 & 0.146608896058 \\ 0.144977134658 & 0.342380788133 & 0.760227774180 & 0.277253172081 \\ 0.065127616053 & 0.146608896058 & 0.277253172081 & 0.665086078202 \end{pmatrix}$	0.41093795738

being a degree n polynomial. Then the Green function has a simple form

$$G_E(x) = \big|\log\big|T(x) + \sqrt{T^2(x) - 1}\big|\big|,$$

which immediately implies that $\mathrm{Cap}(E) = (2|c|)^{-1/n}$.

In Table 2, for several five-segment sets E obtained as the inverse images of a single segment under the map generated by the Chebyshev polynomial $T_5(x)$, we provide the following data: the period matrix of the associated curve $\mathcal{X}$ and the capacity Cap(E) calculated by formula (16). The image of the infinity in the Jacobian remains the same: $u(\infty) = (0.1, 0.2, 0.3, 0.4)$, and the two formulas for the capacity agree at least within an accuracy of 10^{-12}.

5. CONCLUSION

We suggest a compact closed formula for the capacity of the set E of $g + 1$ aligned segments (1):

$$\mathrm{Cap}(E) = (e_{2g+2} - e_1)\left|\frac{\theta[\epsilon, \epsilon'](u(\infty_+))\,\theta[\xi, \xi'](u(\infty_+))}{\theta[\epsilon, \epsilon'](2u(\infty_+))\,\theta[\xi, \xi'](0)}\right|^2, \tag{17}$$

where θ is a genus g theta function with (nonsingular) odd characteristic $[\epsilon, \epsilon']$ and even characteristic $[\xi, \xi']$. Since one can easily deduce how the Green function changes under a conformal mapping, the capacities of the sets with conformally equivalent complements are connected in a predictable way. Grötzsch's uniformization lemma states that any multiply connected domain on a Riemann sphere is conformally equivalent to a parallel-slit domain (that is, the complement of several parallel segments), so in fact our result may yield a closed formula for a whole family of domains with an axial symmetry. Similar formulas for the capacity of condensers of more sophisticated shape will be obtained by the authors in their further publications (see [8]).

ACKNOWLEDGMENTS

The authors are deeply indebted to an anonymous reviewer for pointing out the works [2, 14], where other approaches to the problem of calculating the capacity of a union of segments in terms of function theory on complex curves were discussed.

This work is supported by the Russian Science Foundation under grant 16-11-10349.

REFERENCES

1. N. Achieser, "Sur les polynomes de Tchebyscheff pour deux segments," C. R. Acad. Sci. Paris **191**, 754–756 (1930).
2. A. I. Aptekarev, "Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda lattices," Mat. Sb. **125** (2), 231–258 (1984) [Math. USSR, Sb. **53** (1), 233–260 (1986)].
3. D. Betsakos, K. Samuelsson, and M. Vuorinen, "The computation of capacity of planar condensers," Publ. Inst. Math. **75**, 233–252 (2004).
4. A. B. Bogatyrev, "Elementary construction of Jenkins–Strebel differentials," Mat. Zametki **91** (1), 143–146 (2012) [Math. Notes **91**, 135–138 (2012)].
5. A. B. Bogatyrev, "Conformal mapping of rectangular heptagons," Mat. Sb. **203** (12), 35–56 (2012) [Sb. Math. **203**, 1715–1735 (2012)].
6. A. Bogatyrev, *Extremal Polynomials and Riemann Surfaces* (Springer, Berlin, 2012), Springer Monogr. Math.
7. A. B. Bogatyrev, "Image of Abel–Jacobi map for hyperelliptic genus 3 and 4 curves," J. Approx. Theory **191**, 38–45 (2015).
8. A. B. Bogatyrev and O. A. Grigor'ev, "Conformal mapping of rectangular heptagons. II," Comput. Methods Funct. Theory (in press).
9. V. N. Dubinin and D. B. Karp, "Generalized condensers and distortion theorems for conformal mappings of planar domains," in *The Interaction of Analysis and Geometry* (Am. Math. Soc., Providence, RI, 2007), Contemp. Math. **424**, pp. 33–51.
10. T. Falliero and A. Sebbar, "Capacité d'une union de trois intervalles et fonctions thêta de genre 2," J. Math. Pures Appl. **80** (4), 409–443 (2001).
11. H. M. Farkas and I. Kra, *Riemann Surfaces* (Springer, New York, 1980), Grad. Texts Math. **71**.
12. G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable* (Am. Math. Soc., Providence, RI, 1969), Transl. Math. Monogr. **26**.
13. O. A. Grigor'ev, "Numerical–analytical method for conformal mapping of polygons with six right angles," Zh. Vychisl. Mat. Mat. Fiz. **53** (10), 1629–1638 (2013) [Comput. Math. Math. Phys. **53**, 1447–1456 (2013)].
14. A. L. Lukashov and F. Peherstorfer, "Automorphic orthogonal and extremal polynomials," Can. J. Math. **55**, 576–608 (2003).
15. D. Mumford, *Tata Lectures on Theta* (Birkhaüser, Boston, 1983, 1984, 1991), Vol. II.
16. H. E. Rauch and H. M. Farkas, *Theta Functions with Applications to Riemann Surfaces* (Williams & Wilkins Co., Baltimore, 1974).
17. R. S. Rumely, *Capacity Theory on Algebraic Curves* (Springer, Berlin, 1989), Lect. Notes Math. **1378**.
18. H. Widom, "Extremal polynomials associated with a system of curves in the complex plane," Adv. Math. **3** (2), 127–232 (1969).

Translated by A. B. Bogatyrev