

REAL MEROMORPHIC DIFFERENTIALS: A LANGUAGE FOR DESCRIBING MERON CONFIGURATIONS IN PLANAR MAGNETIC NANOELEMENTS

A. B. Bogatyrev*

We use the language of real meromorphic differentials from the theory of Klein surfaces to describe the metastable states of multiply connected planar ferromagnetic nanoelements that minimize the exchange energy and have no side magnetic charges. These solutions still have sufficient internal degrees of freedom, which can be used as Ritz parameters to minimize other contributions to the total energy or as slow dynamical variables in the adiabatic approximation. The nontrivial topology of the magnet itself leads to several effects first described for the annulus and observed in the experiment. We explain the connection between the numbers of topological singularities of various types in the magnet and the constraints on the positions of these singularities following from the Abel theorem. Using multivalued Prym differentials leads to new meron configurations that were not considered in the classic work by Gross.

Keywords: spintronics, planar nanoelement, magnetic vortex, meron, Klein surface, Prym differential

DOI: 10.1134/S0040577917100117

1. Introduction

Ferromagnets at the nanoscale contain an intriguing world of interacting quasiparticles, the vortexlike patterns of magnetization distribution in the material. The concepts of skyrmions, instantons, and merons were introduced many years ago in high energy physics, but they are also used in condensed matter physics where vortexlike topologies, for example, describe systems with a quantum Hall effect, certain liquid-crystal phases, Bose–Einstein condensates, etc. Magnetic vortices are very important in the emerging spintronics industry, where they are regarded as prospective candidates for information transmission and storage because they are only a few nanometers in size, are very stable, and are easily manipulated with a small energy consumption. With the industrial applications considered, flat configurations of magnets become especially important.

The Hamiltonian governing the dynamics of a magnetic state involves rather many interactions (such as the exchange interaction, the magnetic dipole interaction, the Dzyaloshinskii–Moriya interaction, the interaction with an external magnetic field, the magnetostatic energy, etc.), which form a certain hierarchy. There is an understanding among physicists that for nanoscale planar ferromagnets, the exchange interaction gives the largest contribution to the behavior of a state. One approach for studying magnetic states (see [1]

*Moscow Institute of Physics and Technology, Moscow, Russia; Moscow State University, Moscow, Russia; Institute for Numerical Mathematics, RAS, Moscow, Russia, e-mail: ab.bogatyrev@gmail.com.

This research was supported by a grant from the Russian Science Foundation (Project No. 16-11-10349).

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 193, No. 1, pp. 162–176, October, 2017. Original article submitted November 10, 2016.

and the references therein) is to minimize the Heisenberg exchange energy and thus obtain metastable states whose further dynamics is determined by the Hamiltonian terms next in the hierarchy.

Here, we use the language of real meromorphic differentials from the theory of *Klein surfaces* [2], [3] to describe the metastable states of multiply connected planar ferromagnetic nanoelements minimizing the exchange energy and having no side magnetic charges. The last property means that the corresponding part of the magnetostatic energy is minimized. These solutions still have sufficient internal degrees of freedom, which can be used as Ritz parameters to minimize subsequent terms in the energy hierarchy or as slow dynamical variables in the adiabatic approximation. The nontrivial topology of the magnet itself leads to several effects first described for the annulus [4] and observed in experiments. Here, we explain the topological constraints on the number of magnetization singularities of various types and also the algebraic constraints on their positions following from the Abel theorem.

Using multivalued *Prym differentials* leads to new meron configurations, which were not considered in the fundamental work of Gross [5].

2. Setting the problem

In this section, we briefly formulate the Metlov model describing magnetization textures in planar nanomagnets proposed in [1], [6]. The model is based on assuming a certain hierarchy of different contributions to the total energy of a magnet (such as the exchange energy and the magnetostatic interaction). Such a hierarchy usually holds in magnets of submicron sizes (magnetic nanoelements). In a planar nanomagnet with the shape of a thin cylinder, the magnetization texture in the absence of an external magnetic field can be described using a meromorphic function of a complex variable, which arises in solving the Riemann–Hilbert boundary value problem [7]. This problem can be naturally reformulated in terms of real meromorphic differentials, which are the main subject here.

The exchange energy of a two-dimensional nanomagnet filling a planar domain Ω has the form [8]

$$E[m] = \int_{\Omega} \sum_{a=1}^3 |\nabla m_a(x_1, x_2)|^2 d\Omega, \quad (1)$$

where the (normalized) magnetization distribution $m(x_1, x_2) = (m_1, m_2, m_3)$ is a smooth map from Ω to the unit sphere: $|m|^2 := \sum_a m_a^2 = 1$. Because both the domain and range of the magnetization map m are surfaces, the temptation to use the language of complex variables for this problem arises. For this, we introduce the complex coordinate $z := x_1 + ix_2$ ranging in Ω and also the stereographic projection of the sphere to the complex plane of the variable w :

$$m_1 + im_2 = \frac{2w}{1 + w\bar{w}}, \quad m_3 = \frac{1 - w\bar{w}}{1 + w\bar{w}}. \quad (2)$$

The exchange energy can be expressed in new terms as

$$E[w] = \int_{\Omega} \frac{8}{(1 + w\bar{w})^2} (|w_z|^2 + |\bar{w}_z|^2) d\Omega, \quad (3)$$

where we use the Wirtinger notation for complex differentiation $\partial/\partial z$. We note that the function $w(z)$ can be neither analytic nor antianalytic. The Euler–Lagrange equation for the last functional becomes

$$w_{z\bar{z}} = \frac{2\bar{w}}{1 + w\bar{w}} w_z w_{\bar{z}}. \quad (4)$$

The general solution of this equation is unknown (at least to us). Two wide classes of local solutions are the instantons $w = f(z)$ introduced by Belavin and Polyakov [9] and merons (or half-instantons)

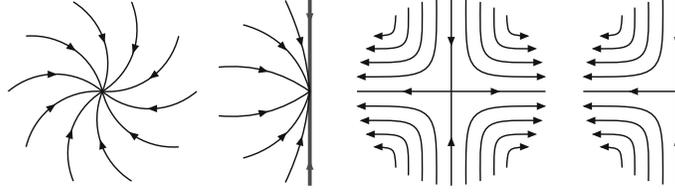


Fig. 1. Simple vortices (corresponding to poles of $d\xi$) and antivortices (or saddles, zeros of $d\xi$) inside the domain and on its boundary.

$w = f(z)/|f(z)|$ discovered by Gross [5]. Here, $f(z)$ is an (anti)holomorphic function of z . It is easy to show that the general solution of (4) with the magnetization m in the plane of the magnet is exactly the meron. Indeed, in this case, the magnetization has the form $m(z) = (\cos \Phi, \sin \Phi, 0)$, and exchange energy (1) simply becomes a Dirichlet integral for the real-valued function $\Phi(z)$:

$$E[m] = \int_{\Omega} (\sin^2 \Phi + \cos^2 \Phi) |\nabla \Phi|^2 d\Omega.$$

This means that $\Phi(z)$ is harmonic and is locally the imaginary part of a holomorphic function $h(z)$. We thus (locally) obtain a meronic representation of $w(z)$ with $f = e^{h(z)}$.

Hence, if any meromorphic function f is given in the domain, then we can construct either an instanton or a meron. We note that a meron corresponding to $f(z)$ with zeros or poles has an infinite energy. In this respect, Gross wrote [5] that “infinite action configurations may be of physical relevance if their action only diverges logarithmically with the volume.” In our case, the energy diverges as the logarithm of the volume of a small vicinity of a zero or pole. Given a meromorphic function f , a mixture of two basic solutions with a finite energy is also considered [1], [5], such as

$$w(z) = \begin{cases} f(z)/E_1, & |f(z)| \leq E_1, \\ f(z)/|f(z)|, & E_1 \leq |f(z)| \leq E_2, \\ f(z)/E_2, & E_2 \leq |f(z)|, \end{cases}$$

or

$$w(z) = \begin{cases} f(z)/E_1, & |f(z)| \leq E_1, \\ f(z)/|f(z)|, & E_1 \leq |f(z)| \leq E_2, \\ E_2/\overline{f(z)}, & E_2 \leq |f(z)|. \end{cases}$$

A solution of such a form is continuous but not smooth: the normal derivative breaks along the lines $|f(z)| = E_1$, $|f(z)| = E_2$, and expression (4) evaluated on such composite solutions acquires sources concentrated on the lines.

To obtain a physically meaningful global solution, we must equip it with an appropriate boundary condition. The impermeability condition—the vector $w(z)$ is (almost everywhere) parallel to the boundary [1], [4], [10]—does not conflict with Eq. (4) and models a domain wall on the boundary. From the mathematical standpoint, this gives a homogeneous Riemann–Hilbert problem [7]. From the physical standpoint, we eliminate side magnetic charges [8] and therefore their magnetostatic energy. We note that minimizing exchange energy functional (3) requires different boundary conditions, namely, the Neumann conditions $w_z dz - w_{\bar{z}} d\bar{z} = 0$ at the boundary.

Summarizing the foregoing, we can extract a reasonable mathematical problem setting: find functions f meromorphic in a domain Ω such that the differential $d\xi := dz/f(z)$ is real on the domain boundary.

The posed problem is exactly the problem of describing all *real meromorphic differentials* in the domain. The transition from functions f to the differentials $d\xi$ reveals the conformal invariance of the initial problem. Indeed, a conformal map transfers real differentials from one domain to the other (with the same conformal moduli) and thus preserves not only solutions of the differential equation but also the impermeability boundary condition. Zeros and poles of the differential (which can also be on the magnet boundary) correspond to topological singularities of the magnetization (see Fig. 1). According to Helmholtz, real meromorphic differentials also describe the (locally potential) stationary flows of an ideal fluid in a planar domain with possible sources or sinks, which gives one more connection between hydrodynamics and (micro)magnets.

3. Schottky double of a planar domain and real differentials

Let Ω be a planar domain of finite connectivity with smooth boundaries. Its Schottky double is a closed Riemann surface of genus one less than the connectivity of Ω , is denoted by $\mathcal{D}\Omega$, and is obtained as follows [2]. We take two copies of the domain and attach their boundary components to one another in a natural manner. If z is a (local) complex coordinate on one copy of Ω , then \bar{z} is the (local) complex coordinate on the other. The double $\mathcal{D}\Omega$ admits an *anticonformal involution* τ (also called a *reflection*) consisting in interchanging two copies of the domain. The fixed points of this reflection are exactly the boundaries of the original domain, called *real ovals* of the surface in this setting. The reflection τ naturally acts on the closed contours on the surface and thus splits the 1-homology classes into even (preserved under reflection) and odd (changing their sign) [2], [11]. Another natural action of the involution is the action on the spaces of holomorphic and meromorphic differentials: *a variable change due to the reflection with a subsequent complex conjugation*. This action splits the differentials into real (remaining unchanged) and imaginary (changing the sign under this action) [2], [3]. Real differentials are exactly those whose restriction to any real oval is real. It can be easily verified that integrating real or imaginary differentials over even cycles respectively gives a real or imaginary answer, while a pairing with odd cycles in contrast is respectively imaginary or real [11].

Constructing all real differentials is in a sense a trivial problem: we take any meromorphic differential on the double $\mathcal{D}\Omega$ and symmetrize it with respect to the induced action of the reflection τ .

3.1. Examples.

Example 1. Real meromorphic differentials in a doubly connected domain can be enumerated as follows. Any topological annulus is conformally equivalent to the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of the variable z with two symmetric real slits, for example, $\pm[1, 1/k]$, where $0 < k < 1$ is the modulus of the domain. The double of this domain is equivalent to the elliptic curve

$$(z, w) \in \mathbb{C}^2 : \quad w^2 = (z^2 - 1)(k^2 z^2 - 1)$$

with the anticonformal involution $\tau(z, w) = (\bar{z}, -\bar{w})$. A meromorphic differential on this surface has the form

$$d\omega(z, w) = (R_1(z) + wR_2(z)) dz \tag{5}$$

with rational functions R_1 and R_2 of z . Among them, real differentials are exactly those satisfying $d\omega(\bar{z}, -\bar{w}) = \overline{d\omega(z, w)}$, i.e., with real rational functions $R_1(z)$ and $iR_2(z)$.

Example 2. Three-connected domains can be treated similarly. Any topological pair of pants is conformally equivalent to the complex plane with three disjoint real slits, for example, $\bigcup_{j=1}^3 [e_{2j-1}, e_{2j}]$,

where three of the points $e_1 < e_2 < \dots < e_6$ can be normalized arbitrarily and three others can be regarded as conformal moduli of the pants. The double of this domain is the genus-2 complex curve

$$(z, w) \in \mathbb{C}^2 : \quad w^2 = \prod_{j=1}^6 (z - e_j) \quad (6)$$

with the reflection τ as in the preceding example, and real differentials have form (5) with real rational functions $R_1(z)$ and $iR_2(z)$ just as in the preceding example.

Example 3. Real differentials can also be obtained in terms of theta functions. We represent an annulus $1 \leq |z| \leq R$ as a quotient of the vertical strip $0 \leq \operatorname{Re}(u) \leq 1/2$ by the group of translations generated by iT , where $T = \pi/\log(R) > 0$. The correspondence between the two models of the annulus is given by the explicit formula $z(u) = e^{2\pi u/T}$.

In the strip model, we can construct all real meromorphic differentials $d\xi$ invariant under the shifts $u \rightarrow u + iT$. The simplest of them is $d\xi = i du$. All others are obtained by multiplication by a meromorphic function $h(u)$ that is real on the strip boundaries and has the needed shift invariance. The symmetry principle extends such a function to a meromorphic function on the entire plane, where it has a period lattice $\mathbb{Z} + iT\mathbb{Z}$. Doubly periodic functions can be effectively represented in terms of elliptic theta functions [12]. We choose a representative a_s^+ (modulo the period lattice) of each zero of the function h and a representative a_s^- of each of its poles. They satisfy the well-known lattice condition [12]

$$\sum_{s=1}^N (a_s^+ - a_s^-) \in \mathbb{Z} + miT, \quad m \in \mathbb{Z}, \quad N = \deg h(u). \quad (7)$$

With this choice of representatives, the function is proportional to

$$h(u) = e^{-2\pi i m u} \prod_{j=1}^N \frac{\theta_1(u - a_j^+)}{\theta_1(u - a_j^-)}, \quad (8)$$

where $\theta_1(u) = e^{-\pi T/4} \sin(\pi u) - \dots$ is the only odd theta function of modulus iT .

If the sets of zeros and poles of $h(u)$ are mirror symmetric with respect to the imaginary axis, then we can deduce from the symmetries of the theta function that the (suitably normalized) function $h(u)$ is real on the strip boundaries $\{0, 1/2\} + i\mathbb{R}$. We transfer the differential $d\xi = ih(u) du$ with poles to the concentric ring using the exponential map $z(u)$, and lattice condition (7) then becomes the necessary and sufficient condition for the existence of a real differential with the given sets of zeros and poles in the annulus:

1. The number of zeros equals the number of poles.
2. The total number of zeros and poles on each boundary (half-vortices) is integer.
3. The sum of the azimuths of poles is equal to the sum of the azimuths of zeros modulo π .

Here, the azimuth is measured with respect to the center of the annulus modulo 2π , and zeros and poles on the boundary are counted with the weight $1/2$.

The authors of [4] checked the validity of the last constraint on the positions of magnetic vortices in nanoelements with the available experimental data and found a good agreement with the experiment.

3.2. Homologies and differentials on the Schottky double. From the last example, we learn that the positions of zeros and poles of a real differential in an annulus cannot be arbitrary. The same effect is observed in domains with greater connectivity, and it is essentially related to the Abel theorem on the divisors of algebraic functions [13], [14]. To explain this effect, we introduce some auxiliary constructions.

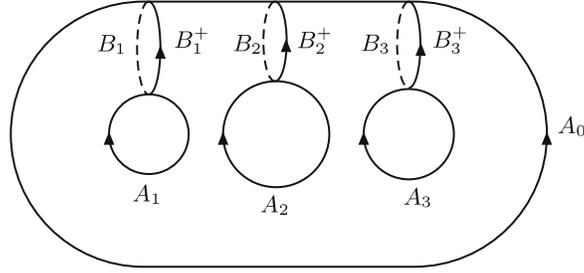


Fig. 2. Homology basis on the double $\mathcal{D}\Omega$ of the flat domain Ω .

3.2.1. Even and odd basic cycles. On the double of a $(g+1)$ -connected domain Ω , we introduce a basis $A_1, \dots, A_g; B_1, \dots, B_g$ of integer 1-homologies (closed contours modulo a certain equivalence) with the standard symplectic intersection form

$$A_j \circ A_s = B_j \circ B_s = 0, \quad A_j \circ B_s = \delta_{js}, \quad j, s = 1, \dots, g, \quad (9)$$

and the additional mirror symmetry

$$\tau A_j = A_j, \quad \tau B_j = -B_j, \quad j = 1, \dots, g. \quad (10)$$

A basis with the above properties is not unique. For instance, we can take the interior boundaries of $\Omega \subset \mathcal{D}\Omega$ with the induced orientation as A cycles. We connect each of them to the exterior boundary $A_0 \sim -\sum_{j=1}^g A_j$ with g pairwise disjoint simple arcs B_j^+ , which we close to odd cycles on the double: $B_j := B_j^+ - \tau B_j^+$ (see Fig. 2).

3.2.2. Holomorphic differentials. Holomorphic and meromorphic differentials are used, for example, to represent various functional-theoretical objects on Riemann surfaces. We construct a basis of holomorphic differentials. There are g so-called harmonic measures $W_s(z)$ in the domain Ω , i.e., harmonic functions vanishing at all the boundaries except the one interior boundary A_s , where it is equal to unity. The conjugate harmonic functions $H_s(z)$ are multivalued in the domain, but the differentials $d\omega_s := d(W_s + iH_s)$ are holomorphic and purely imaginary on the boundaries. This allows holomorphically extending them to the closed surface $\mathcal{D}\Omega$ as imaginary differentials:

$$\tau \cdot d\omega_s = -d\omega_s, \quad s = 1, \dots, g.$$

The periods of these differentials are

$$\int_{B_s} d\omega_j = -2\delta_{js}, \quad \int_{A_s} d\omega_j = i\Pi_{js},$$

where Π_{js} is a symmetric real matrix with a positive diagonal, negative off-diagonal elements, and diagonal dominance. It can be interpreted as the capacity matrix of the domain. Indeed, W_j can be regarded as a voltage that induces the charge

$$\int_{A_s} \left(\frac{dW_j}{dn} \right) dl = \int_{A_s} \left(\frac{dH_j}{dl} \right) dl = -i \int_{A_s} d\omega_j = \Pi_{js}$$

on the boundary component A_s (n and l are the directions normal and tangent to the boundary).

Real differentials $d\zeta := (d\zeta_1, \dots, d\zeta_g)^t$ with the usual A -normalization

$$\int_{A_s} d\zeta_j = \delta_{js}$$

are related to the imaginary differentials $d\omega := (d\omega_1, \dots, d\omega_g)^t$ as

$$i\Pi d\zeta = d\omega.$$

3.2.3. Meromorphic differentials. To construct magnetization distributions with singularities, we need differentials with simple poles. Let W_{pq} be a harmonic function in the domain Ω with exactly two logarithmic singularities of opposite signs at points p from the closure of the domain Ω and q from its interior. This function satisfies the homogeneous Neumann boundary condition

$$W_{pq}(z) = m(p) \log |z - p| - \log |z - q| + \text{harmonic function}, \quad \left. \frac{\partial W_{pq}(z)}{\partial n} \right|_{z \in \partial\Omega} = 0,$$

$$m(p) = \begin{cases} 1, & p \in \text{Int } \Omega, \\ 2, & p \in \partial\Omega. \end{cases}$$

Let $H_{pq}(z)$ be a harmonic function (multivalued in the domain) conjugate to $W_{pq}(z)$. Then the differential $d\eta_{pq} := d(W_{pq} + iH_{pq})$ is single-valued and real on the boundary of Ω . It can be extended meromorphically to the closed surface $\mathcal{D}\Omega$ using the reflection action as a real differential,

$$\tau \cdot d\eta_{pq} = d\eta_{pq},$$

which satisfies the A -normalization conditions $\int_{A_s} d\eta_{pq} = 0$, $s = 0, \dots, g$ (the integral is taken in the sense of the principal value if p is on the boundary).

3.3. Constraints on the poles and zeroes. Here, we study the constraints on the positions of magnetization singularities in a multiply connected nanoelement described by a real differential that are similar to those that an annulus has.

The reflection τ naturally acts on the divisors, finite formal sums of points $p \in \mathcal{D}\Omega$ with integer coefficients:

$$D = \sum_p c(p) \cdot p \rightarrow \tau D = \sum_p c(p) \cdot \tau p.$$

For each mirror-symmetric divisor $D = \tau \cdot D$, for instance, the divisor $(d\omega)$ of zeros and poles of a real differential $d\omega$, we introduce its *reduction* D/τ as the divisor with the support in the closure of the domain Ω . Interior points of the reduction D/τ inherit their multiplicity from D , and boundary points have only half of that and can thus have a half-integer weight. A symmetric divisor is equal to the sum of its reduced divisor and its reflection. Every divisor D (including reduced divisors) naturally decomposes into positive and negative parts: $D = D^+ - D^-$, $D^\pm \geq 0$.

Theorem 1. *The reduced divisor of a real differential $d\xi$ in Ω is completely characterized by the fact that the divisor $D := (d\xi)/\tau - (d\omega)/\tau =: \sum_p c(p) \cdot p$, where $d\omega \neq 0$ is any fixed real differential, satisfies the three conditions*

$$\deg(D) := \sum_p c(p) = 0, \tag{11}$$

$$\deg(D|A_s) := \sum_{p \in A_s} c(p) \in \mathbb{Z}, \quad s = 0, \dots, g, \tag{12}$$

$$\left(2 \sum_p c(p) \cdot \mathbf{H}(p) := \right) 2 \int_{D^+}^{D^-} d\mathbf{H} \in \Pi\mathbb{Z}^g, \tag{13}$$

where $d\mathbf{H}(z) := (dH_1, \dots, dH_g)^\dagger$ and the integral with divisor limits is the sum of integrals between points with the same weight multiplied by this weight (some interior points can be split in half if necessary).

A real differential with a given reduced divisor is unique up to multiplication by a real constant and is given by the formula

$$d\xi(z) := d\omega(z) \exp\left(\int_{z_0}^z \sum_p c(p) d\eta_{pq}\right), \quad (14)$$

where $z_0 \in A_0$, q is any interior point of the domain Ω , and $c(p)$ is the half-integer multiplicity of the point p in the reduced divisor D satisfying (11)–(13).

Remark 1. Real lattice conditions (13) are independent of the choice of the auxiliary differential $d\omega$, the decomposition of the divisors in the integration limits into pairs of points, and the choice of integration paths connecting them in the domain Ω .

Example 4. We determine the constraints on the positions D of magnetization singularities in a topological pair of pants. Using the conformal invariance of real differentials, we conformally map the pants to the extended plane with three disjoint real slits as in Example 2. The Schottky double of the pants now has representation (6) with the reflection $\tau(z, w) = (\bar{z}, -\bar{w})$. An auxiliary real holomorphic differential in this model has the form $d\omega = (i(z - c)/w) dz$ with real c . As the symplectic basis of cycles, we choose $A_1, A_2; B_1, B_2$, which are the inverse images of the respective segments $[e_1, e_2], [e_5, e_6]; [e_2, e_3], [e_4, e_5]$ under the projection $(z, w) \rightarrow z$ with suitable orientation. The basis of imaginary holomorphic differentials has the form $d\omega_s = (a_s(z - c_s)/w) dz$, where the real coefficient a_s and the zeros $c_1 \in [e_4, e_5]$ and $c_2 \in [e_2, e_3]$ can be found from the normalization conditions $\int_{B_s} d\omega_j = -2\delta_{js}$. Restrictions (13) for the positions D^+ of zeros and the positions D^- of poles of a real differential are now

$$2 \operatorname{Im} \int_{D^+}^{D^-+c} d\omega \in \Pi\mathbb{Z}^2.$$

This condition is independent of the real c : if $w^2(c) > 0$, then an increment of c leads to a real increment of this integral, and if $w^2(c) < 0$, then c represents a pair of half-points on the boundary of the pants, and an increment of c does not change the total integral.

Proof of Theorem 1 [2], [3], [15]. We fix the auxiliary real differential $d\omega$, for example, the holomorphic differential $d\zeta_s$ or $i d\omega_s$. The ratio of two real differentials is a real function: $h(z) = d\xi(z)/d\omega(z)$. We let $D = \sum_p c(p) \cdot p$ denote the reduced divisor of $h(z)$ and verify the three properties (11)–(13). The first two are obvious: $0 = \deg(h) = 2 \deg D$. On each boundary component, the real function $h(u)$ changes its sign at an even number of points p , its zeros and poles of odd degree, i.e., exactly where $c(p) \in 1/2 + \mathbb{Z}$.

We now concentrate on (13). For every interior point $q \in \Omega$, we have the representation

$$\frac{dh}{h} = \sum_p c(p) d\eta_{pq},$$

where $c(p) \in \mathbb{Z}/2$ is the multiplicity of the point p in the reduced divisor of $h(z)$. Indeed, both sides have the same singularities, and the integrals over each boundary component A_s (understood, perhaps, in the sense of the principal value) vanish. We now have a chain of vector equalities, where \mathbf{B} denotes the set of B -cycles on the surface $\mathcal{D}\Omega$:

$$\begin{aligned} 2\pi i \mathbb{Z}^g \ni \int_{\mathbf{B}} \frac{dh}{h} &= \sum_p c(p) \int_{\mathbf{B}} d\eta_{pq} \stackrel{*}{=} 2\pi i \sum_p 2c(p) \operatorname{Re} \left[\int_q^p d\zeta \right] = \\ &= 2\pi i \sum_p 2c(p) \operatorname{Im} \left[\Pi^{-1} \int_q^p d\omega \right] = 2\pi i \sum_p 2c(p) \Pi^{-1} \int_q^p d\mathbf{H}, \end{aligned} \quad (15)$$

where the equals sign with the asterisk follows from the Riemann bilinear relations (reciprocity law) and all the sums over p are taken over the support of the reduced divisor D . We group the terms with the opposite value of $c(p)$ in last sum and obtain exactly condition (13).

Converse argument. Let the three conditions in the theorem hold for the reduced divisor D . We must verify three statements for differential (14):

1. The differential $d\xi$ in the domain of the reduced divisor is exactly $D + (d\omega)/\tau$. Indeed, $\deg D = 0$, and the integral in (14) has no singularity at $z = q$.
2. The differential $d\xi$ is single-valued in the domain. This follows from the A -normalization of the elementary singular differentials $d\eta_{pq}$ and condition (12).
3. The differential is real on the boundaries of the domain. Indeed,

$$2i \operatorname{Im} \left[\int_{B_s^+} d\eta_{pq} \right] = \int_{B_s} d\eta_{pq}.$$

The sum of these integrals over the points p taken with their weights can be extracted from chain of equalities (15), which together with restriction (13) means that the exponential factor in (14) is real on the boundary of the domain Ω .

4. Real Prym differentials and nonlocal merons

Gross writes [5]: “Even though we have found an infinite class of solutions of the nonlinear equations of motions, it is clear that we do not have the most general solution.” We describe all the omitted meronic solutions in this section. We recall (Sec. 2) that the general local solution of (4) with $|w(z)| = 1$ outside the magnetization singularities has the form $w(z) = f/|f|$ with a holomorphic function $f(z) \neq 0$. This function can even be non-single-valued globally and yet lead to a single-valued magnetization iff $f(z)$ is multiplied by a positive function in bypassing the singularities and/or boundary components of the magnet. This positive function is necessarily (locally) constant, which leads to the appearance of a monodromy:

$$\rho: \pi_1(\Omega - \operatorname{Supp}(f)) \rightarrow \mathbb{R}^+. \quad (16)$$

A (commutative) monodromy in a finitely punctured domain is completely determined by the monodromies $\rho(p)$ of loops counterclockwise encircling punctures $p \in \operatorname{Supp}(f)$ and the monodromies $\rho(A_s)$ of oriented boundary components with the only relation

$$\prod_p \rho(p) = \prod_{s=0}^g \rho(A_s).$$

We determine the form of $f(z)$ near an isolated magnetization point singularity $z = p$. The function $h(z) := (z - p)^{-i\mu}$, $2\pi\mu = \log(\rho(p))$, has the same local monodromy as does f , and this means that the function f/h is single-valued around $z = p$. This point cannot be an essential singularity of f/h because we otherwise lose the property of the logarithmic divergence of the energy (see Sec. 2), but it can be a pole or a zero of finite order. We thus obtain the local behavior of the multivalued differential $d\xi := dz/f(z)$ near points of its divisor:

$$d\xi(z) = (z - p)^{m+i\mu} \phi(z) dz, \quad \phi(z) \in \mathcal{O}^*(p), \quad m \in \mathbb{Z}, \quad \mu \in \mathbb{R}, \quad (17)$$

with a holomorphic function $\phi(z)$ that is nonvanishing in the vicinity of the point p . This differential has a positive monodromy and, just as before, is real on the domain boundary. The last property is well defined even with the monodromy taken into account. Such objects are called real Prym differentials [15], [16].

The divisor $D = (d\xi)$ of a real Prym differential $d\xi$ has complex weights $c(p) = m(p) + i\mu(p)$ with a real part (integer in our case) determined by the increase of the differential near the singularity $z = p$ and an imaginary part related to the positive local monodromy. This divisor is unchanged under the natural action of the reflection τ on divisors with complex weights:

$$D := \sum_p c(p) \cdot p \rightarrow \tau \cdot D := \sum_p \overline{c(p)} \cdot \tau p.$$

In particular, points on the real ovals have a purely real weight. For a symmetric complex-weighted divisor $D = \tau \cdot D$ on the double, we define its reduction D/τ to the domain Ω of the double as a divisor supported in the closure of the domain with preservation of the multiplicities of interior points and half of multiplicities of boundary points.

Theorem 2. *The reduced divisor of a real Prym differential $d\xi$ in Ω with a given monodromy $\rho: \pi_1(\Omega - \text{Supp}(d\xi)) \rightarrow \mathbb{R}^+$ is completely characterized by the fact that the divisor with the complex weights $D := (d\xi)/\tau - (d\omega)/\tau := \sum_p c(p) \cdot p$, where $d\omega \neq 0$ is any fixed real Abelian differential, satisfies the four conditions that*

1. *the imaginary part of D is determined by the monodromy,*

$$\text{Im } D = -\frac{1}{2\pi} \sum_p \log \rho(p) \cdot p,$$

2. *$\deg \text{Re } D = 0$, the weights $\text{Re } c(p)$ are integer inside the domain and half-integer on the boundary,*
3. *$\deg \text{Re}(D|_{A_s}) := \sum_{p \in A_s} \text{Re } c(p) \in \mathbb{Z}$, $s = 0, \dots, g$, and*
4. *the lattice condition*

$$2 \text{Im} \sum_p c(p) \int_{z_0}^p d\omega + \frac{1}{\pi} \log \rho(\mathbf{A}) \in \Pi \mathbb{Z}^g \quad (18)$$

holds, where $\log \rho(\mathbf{A})$ is the vector $(\log \rho(A_1), \dots, \log \rho(A_g))^t$ and $z_0 \in A_0$.

A real Prym differential is reconstructed from its divisor uniquely up to multiplication by a real constant:

$$d\xi(z) = d\omega(z) \exp\left(\int_{z_0}^z d\eta + \sum_{s=1}^g \log \rho(A_s) d\zeta_s\right), \quad (19)$$

where $d\eta$ is a (real Abelian) A -normalized differential on the double $\mathcal{D}\Omega$ with only simple poles and the residue divisor $D + \tau \cdot D$, where the reduced divisor D satisfies the above four conditions.

Proof. The proof of this theorem more or less follows the arguments for Theorem 1. The ratio $h(z) = d\xi/d\omega$ is a real Prym function with the same monodromy as $d\xi$ and the reduced divisor D . The real single-valued differential $d \log h(z)$ can be decomposed as

$$d \log h = d\eta + d\zeta$$

where the A -normalized differential $d\eta$ has the same singularities as $d \log h$ (it now cannot be decomposed into the elementary terms $d\eta_{pq}$ defined in Sec. 3.2.3; we use the existence theorem [15]) and the real

holomorphic differential $d\zeta$ whose further decomposition into the basic differentials $d\zeta_s$ can be found by integrating both sides of the last equality along the real ovals. The inclusion

$$2\pi i\mathbb{Z}^g \ni 2 \operatorname{Im} \left[\int_{\mathbf{B}^+} d \log h \right] = \int_{\mathbf{B}} d \log h$$

after substituting the above decomposition and using the Riemann bilinear relations finally gives lattice condition (18) in the theorem.

4.1. Examples.

4.1.1. Disc. We list all nonlocal meron solutions in a simply connected domain. Such domains can be conformally mapped to a half-plane, and we assume that $\Omega = \mathbb{H}$. The general real Prym differential has the form $d\xi(z) = h(z) dz$, where the real function $h(z)$ is the product of several elementary functions of the form

$$\begin{aligned} h^\pm(z | a, \mu) &= (z - a)^{\pm 1 + i\mu} (z - \bar{a})^{\pm 1 - i\mu}, \quad a \in \mathbb{H}, \quad \mu \in \mathbb{R}, \\ h^\pm(z | c) &= (z - c)^{\pm 1}, \quad c \in \mathbb{R}, \end{aligned}$$

with various parameters \pm , a , μ , and c .

4.1.2. Annulus. In terms of elliptic theta functions, we here list all nonlocal meron solutions in the annulus represented as a quotient of the strip $0 \leq \operatorname{Re} u < 1/2$ by the group of shifts. Such a model was introduced in Example 3 (see Sec. 3.1). The map to an annulus is given by the exponential function $z(u) = e^{2\pi u/T}$, $T > 0$. The general solution $d\xi(u) = ih(u) du$ can always be decomposed into elementary solutions: (1) without singularities (holomorphic), (2) interior vortex and antivortex, and (3) one interior antivortex and two half vortices on the boundary. These three cases correspond to the following choices of the real meromorphic function $h(u)$ in the strip:

1. A solution in an annulus without singularities corresponds to

$$h(u | m) = e^{2\pi i m u}, \quad m \in \mathbb{Z}.$$

In this case, the integral curves of the magnetization field $w(z)$ comprise $|m|$ embedded Reeb foliations in the annulus with interior magnetic domain walls, the concentric limit cycles of the magnetization field. Patterns of this type have been observed in experiments.

2. A solution with a vortex and a saddle (antivortex) at two interior points of the annulus corresponds to

$$h(u | a_\pm, \mu_\pm) = \frac{\theta_1^{1+i\mu_+}(u - a_+) \theta_1^{1-i\mu_+}(u + \bar{a}_+)}{\theta_1^{1+i\mu_-}(u - a_-) \theta_1^{1-i\mu_-}(u + \bar{a}_-)}, \quad \mu_\pm \in \mathbb{R}, \quad 0 < \operatorname{Re} a_\pm < \frac{1}{2}.$$

The reduced divisor $D = (1 + i\mu_+) \cdot z(a_+) - (1 + i\mu_-) \cdot z(a_-)$ in the annulus determines the monodromy of the solution.

3. The magnetization configuration contains an antivortex in the interior of the annulus and two half-vortices on its boundary for

$$\begin{aligned} h(u | a, a_1, a_2, \mu) &= \frac{\theta_1^{1+i\mu}(u - a) \theta_1^{1-i\mu}(u + \bar{a})}{\theta_1(u - a_1) \theta_1(u - a_2)}, \\ \mu \in \mathbb{R}, \quad \operatorname{Re} a_1 = \operatorname{Re} a_2 &\in \left\{ 0, \frac{1}{2} \right\}, \quad 0 < \operatorname{Re} a < \frac{1}{2}. \end{aligned}$$

The reduced divisor of the real Prym differential $d\xi = ih(u) du$ transferred to the annulus is

$$D = (1 + i\mu) \cdot z(a) - \frac{1}{2} \cdot z(a_1) - \frac{1}{2} \cdot z(a_2).$$

In all three cases, we verify that the function $h(u)$ is 1-periodic, real on the strip boundaries, and acquires a constant positive factor under the translation $u \rightarrow u + iT$.

5. Conclusion

This paper has been devoted to the mathematical aspects of describing topologically charged magnetization textures [1], [6] in multiply connected planar magnetic nanoelements. New multivortex configurations, exact solutions of the nonlinear boundary value problem describing micromagnetic states in the Metlov model, were introduced in [10] in terms of so-called real meromorphic differentials on a closed Riemann surface, the Schottky double of the original domain. This approach emphasizes the conformal invariance of the problem and allows giving effective formulas for solutions in algebraic terms and also in terms of theta functions [12], [14], Schottky functions [17], and the Schottky–Klein prime form [10], [14], [18]. This approach allowed explicitly listing all magnetization textures following from the model [1], [6] in one-, two-, and three-connected domains. We established the complete set of constraints on the numbers and positions of the magnetization singularities in an arbitrary multiply connected domain. This extends our previous result for the constraints on the positions of topological singularities in an annulus [4], which were verified in the experiment. The constraints on the positions of magnetization singularities can be interesting for applications: in spintronics, vortex configurations are carriers of information, and the possibility arises of verifying whether reading errors occurred.

In Sec. 4, we used the property of a meron field of magnetization to remain unchanged under stretching of its generating meromorphic function. Mathematically, this leads to nonlocal objects, Prym differentials on the double of a planar domain. Those solutions have new hidden degrees of freedom, the local and global monodromies, which can be used as Ritz parameters to minimize the next terms in the energy hierarchy or as slow dynamical variables in the adiabatic approach.

Acknowledgments. The author is indebted to Konstantin Metlov of the Donetsk Institute of Physics and Technology for the introduction to the magic world of magnetism and for setting physically significant mathematical problems.

REFERENCES

1. K. L. Metlov, “Magnetization patterns in ferromagnetic nanoelements as functions of complex variable,” *Phys. Rev. Lett.*, **105**, 107201 (2010).
2. M. Schiffer and D. C. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton Univ. Press, Princeton, N. J. (1954).
3. N. L. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces* (Lect. Notes Math., Vol. 219), Springer, Berlin (1971).
4. A. B. Bogatyrev and K. L. Metlov, “Topological constraints on positions of magnetic solitons in multiply-connected planar magnetic nanoelements,” *Phys. Rev. B*, **95**, 024403 (2017); arXiv:1609.02509v1 [cond-mat.mes-hall] (2016).
5. D. J. Gross, “Meron configurations in the two-dimensional $O(3)$ σ -model,” *Nucl. Phys. B*, **132**, 439–456 (1978).
6. K. L. Metlov, “Two-dimensional topological solitons in soft ferromagnetic cylinders,” arXiv:cond-mat/0102311v2 (2001).
7. F. D. Gakhov, *Boundary Value Problems* [in Russian], Nauka, Moscow (1977); English transl. prev. ed. (Intl. Series Monogr. Pure Appl. Math., Vol. 85), Pergamon, Oxford (1966).

8. A. Aharoni, *Introduction to the Theory of Ferromagnetism*, Oxford Univ. Press, Oxford (1996).
9. A. A. Belavin and A. M. Polyakov, “Metastable states of two-dimensional isotropic ferromagnets,” *JETP Lett.*, **22**, 245–247 (1975).
10. A. B. Bogatyrev and K. L. Metlov, “Magnetic states in multiply-connected flat nano-elements,” *Low Temp. Phys.*, **41**, 767–771 (2015); arXiv:1504.01162v1 [cond-mat.mes-hall] (2015).
11. A. Bogatyrev, “Elementary construction of Jenkins–Strebel differentials,” *Math. Notices*, **91**, 135–138 (2012).
12. N. I. Akhiezer, *Elements of the Theory of Elliptic Functions* [in Russian], Nauka, Moscow (1970).
13. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York (1994).
14. D. Mumford, *Tata Lectures on Theta* (Progr. Math., Vol. 28), Vols. 1–3, Birkhäuser, Boston (1983).
15. H. M. Farkas and I. Kra, *Riemann Surfaces* (Grad. Texts Math., Vol. 71), Springer, New York, Berlin (1980).
16. V. V. Chueshev, *Multiplicative Functions and Prym Differentials on a Variable Compact Riemann Surface* [in Russian], Kemerovo State Univ., Kemerovo (2003).
17. A. Bogatyrev, “Computations in moduli spaces,” *Computat. Meth. Funct. Theory*, **7**, 309–324 (2007).
18. A. Bogatyrev, “Prime form and Schottky model,” *Computat. Meth. Funct. Theory*, **9**, 47–55 (2009).