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CONSTRUCTIVE APPROXIMATION

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How Many Zolotarëv Fractions are There?

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Abstract Known properties of the Chebyshev polynomials are the following: they have simple critical points and only two (finite) critical values. Those properties uniquely determine the polynomials modulo affine transformations of dependent and independent variables. Zolotarëv fractions have similar properties: simple critical points with only four distinct critical values. These properties determine many classes of rational functions modulo projective transformations of the dependent and independent variables. In this paper, we explicity describe these classes.

Keywords Zolotarev fraction · Elliptic curves · Lattices

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E. I. Zolotarëv has found the best uniform rational approximation for the signum function on two real intervals separated by zero [1,13]. Known today as Zolotarëv fractions, those functions possess many interesting properties that led in particular to their applications in electrical engineering (Cauer or elliptic filters). Despite a rather involved representation (see below), the construction itself of Zolotarëv fractions seems very natural, and it is no wonder that they have been rediscovered several times [6,9,12] since 1877.

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The parametric formulas for classical degree n Zolotarëv fractions $Z_n(x|\tau)$ are given in terms of the elliptic sine $sn(\cdot|\tau)$ and the complete elliptic integral $K(\tau)$, both of modulus $\tau \in i\mathbb{R}_+$ [1,2,4]:

$$Z_n(x_{n\tau}(u)|\tau) := x_\tau(u), \qquad u \in \mathbb{C}, \tag{1}$$

where $x_\tau(u) = sn(K(\tau)u|\tau)$. At first glance, it is not even clear that Z_n is a rational function of x; the same difficulty emerges for Chebyshev polynomials $T_n(cos(u)) = cos(nu)$.

The geometric interpretation (most likely not a new one) for the above formula (1) leads to a natural enumerative problem for the arising construction. The solution to the latter problem is given in this paper.

1 Twisted Zolotarëv Fractions and Their Classes

One of the characteristic features of the above functions Z_n we take as a definition:

Definition We call a rational function $R(x)$ a twisted Zolotarëv fraction if and only if all of its critical points (including those at infinity if any) are simple with exactly four distinct values. Two fractions are in the same class if and only if they differ by pre- and post-compositions with complex linear fractional functions.

Remark (A) The above definition of (twisted) Zolotarëv fractions is not standard, and it is a replica of ($g = 1$) extremal rational functions [3]. Classical fractions (1) surely meet this definition when $n > 2$: their critical points are exactly inner alternation points corresponding to the uniform approximation problem [1] mentioned above.
(B) The global change of dependent and independent variables of a rational function by elements of $PSL_2(\mathbb{C})$ cannot be considered as a significant modification of this function. In particular, it keeps all its discrete invariants intact: the degree, the number of critical points and their indexes, and the monodromy [5]. We shall see below that recently introduced *Elliptic rational functions* [6] and *Chebyshev–Blaschke products* [9] fall into the same class with classical Zolotarëv fractions.
(C)The topic of algebraic (in particular rational) functions with a few distinct critical values, e.g., Belyi/Fried functions with respectively 3/4 critical values, has remained vibrant for decades. Many prominent mathematicians, such as V. Arnold, G. Belyi, and A. Grothendieck, have contributed to this subject. The outline of this and related topics may be found, e.g., in [5,10]. This subject is closely related to discrete objects such as Dessins d'Enfants, ribbon graphs, etc. and contains many arithmetic, combinatorial, and enumerative problems. One of them is considered in this paper, and the answer is obtained in terms of certain arithmetic functions (such as the number of divisors of an integer, the number of its decompositions into two squares, etc.), which is not a surprise.

The sets B of four critical values of twisted Zolotarëv fractions from the same class make up an orbit of the $PSL_2(\mathbb{C})$-action on 4-tuples of points. Those orbits are distinguished by the value of the so-called j-invariant [2,11]:

$$j := \frac{4}{27}\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2},$$

where λ is a cross-ratio of the four points of B and the right-hand side of the expression is the principal invariant of the so-called anharmonic group. The latter interchanges the values of the cross ratio

$$\lambda,\ 1-\lambda,\ \frac{\lambda}{\lambda-1},\ \frac{1}{1-\lambda},\ \frac{1}{\lambda},\ \frac{\lambda-1}{\lambda},$$

under the permutation of points in B.

The purpose of this paper is twofold: (i) to give a simple constructive description for classes of twisted Zolotarëv fractions, and (ii) to find their number given the degree and the (finite) value of the j-invariant.

2 Construction of Twisted Zolotarëv Fractions

Consider the rank two lattice L in the complex plane of variable u; we designate by the same letter L the group of translations of the plane by the elements of the lattice. Let L^+ be the extension of the translation group by the central symmetry: $u \to -u$. The extended group acts discontinuously in the complex plane, so its orbit space is well defined and carries a natural complex structure

$$\mathbb{C}/L^+ = \mathbb{C}P^1.$$

We can introduce a global coordinate on this Riemann sphere, say

$$x_L(u) = \wp(u|L) := u^{-2} + \sum_{0\neq v\in L}((u-v)^{-2} - v^{-2}); \tag{2}$$

transition to any other global coordinate is given by a linear fractional map.

Once we have a full rank sublattice M of L, the group M^+ is a subgroup of L^+ and any orbit of M^+ is contained in the orbit of L^+. Therefore we have a holomorphic mapping from one sphere to the other:

$$\mathbb{C}/M^+ \to \mathbb{C}/L^+, \qquad u \in \mathbb{C},$$

which becomes a rational function once we fix a complex coordinate on each sphere. Thus we obtain a degree $|L:M|$ rational function $R_{L:M}(x)$:

$$R_{L:M}(x_M(u)) := x_L(u), \qquad x_M(u) := \wp(u|M), \qquad u \in \mathbb{C}. \tag{3}$$

Example (A) To get the modulus $\tau \in i\mathbb{R}_+$ classical Zolotarëv fraction, we take $L = L(\tau) := Span_{\mathbb{Z}}\{4, 2\tau\}$ and $M := Span_{\mathbb{Z}}\{4, 2n\tau\} =: L(n\tau)$. Then $R_{L:M}(x)$ and $Z_n(x|\tau)$ lie in the same class; that is, they differ by pre- and post-compositions with linear fractional functions. Indeed, $x_\tau(u+1) := sn(K(\tau)(u+1)|\tau)$ is a degree two even elliptic function of u with the period lattice $L(\tau)$. Therefore it can be taken as a global coordinate in the orbit space of the group $L^+(\tau)$ instead of a Weierstrass function (2). Classical Zolotarëv fractions are therefore described by the construction (3).

(B) Elliptic rational functions $Y_n(x|\tau)$, also dubbed Chebyshev–Blaschke products, are defined as follows [6,9]:

$$Y_n(y_{1,\tau}(u)|\tau) := y_{n,\tau}(u), \qquad u \in \mathbb{C},$$

where $y_{n,\tau}(u) := \sqrt{k(n\tau)}cd(2K(n\tau)nu|n\tau)$. The latter function $y_{n,\tau}$ is a degree two even elliptic function of u with the period lattice $L_n(\tau) := Span_{\mathbb{Z}}\{2/n, \tau\}$ which may be taken as a global coordinate in $\mathbb{C}/L_n^+$. Hence, the function $Y_n(\cdot|\tau)$ fits the description (3) with $L = L_n(\tau)$ and $M = L_1(\tau)$.

(C) When $M = nL$, function (3) coincides with the so-called flexible Lattés map [7] of degree n^2. In general, the construction does not reduce to a Lattés map. For instance, the degree of Zoloterev fraction is not necessarily a full square.

Theorem 1 *Construction* (3) *gives a one-to-one correspondence between classes of twisted Zolotarëv fractions and the pairs of embedded rank two lattices $M \subset L \subset \mathbb{C}$ modulo multiplication by a nonzero complex number, with two exceptions: (i) $|L : M| = 2$ and (ii) $M = 2L$.*

Proof (A) Let $M \subset L$ be two lattices in the complex plane and $R_{L:M}$ be a corresponding rational function (3). Let us check that it has only simple critical points with the values in a 4-element set. Indeed, critical points of $x_L(u)$ correspond to fixed points of the L^+ action on the plane. They are simple and make up the lattice $\frac{1}{2}L$. Critical values of the projection $x_L(u)$ make up a 4-element set since $|\frac{1}{2}L : L| = 4$. Critical points of the function $R_{L:M}$ correspond to the set $\frac{1}{2}L \setminus \frac{1}{2}M$ in the complex plane.

Let us count how many different orbits of L^+ there are in the set $\frac{1}{2}L \setminus \frac{1}{2}M$. One can show (using the Schmitt normal form of integer matrices [8]) that there are bases in the lattice and its sublattice related by an integer diagonal matrix. Without loss of generality, we assume that $L = Span_{\mathbb{Z}}(1, \tau)$ and $M = Span_{\mathbb{Z}}(p, q\tau)$ with positive integers $p \ge q$ and $Im\ \tau \ne 0$. When $p \ge 3$, no one of four points $u = \frac{1}{2}, 1, \frac{1}{2} + \frac{1}{2}\tau, 1 + \frac{1}{2}\tau$ belongs to the lattice $M/2$, and no two of them are in the same orbit of L^+; therefore $R_{L:M}(x)$ has exactly four critical values.

Otherwise, if $p = q = 2$, we have $M = 2L$, and the set $(\frac{1}{2}L \setminus \frac{1}{2}M)/L^+$ has only three points represented, e.g., by $u = \frac{1}{2}, \frac{1}{2} + \frac{1}{2}\tau, \frac{1}{2}\tau$. The corresponding function $R_{L:M}(x)$ falls into the class of one of the simplest Belyi functions: $x^2 + x^{-2}$, the Hauptmodul for the Klein's quadratic group. The remaining case $p = 2, q = 1$ corresponds to the index 2 sublattice, and $R_{L:M}(x)$ has just two critical points as well as values; therefore it is in the class of the quadratic function x^2.

In case we rescale the lattices, the rational function $R_{L:M}$ changes in a predictable way since

$$\wp(cu|cL) = c^{-2}\wp(u, L), \qquad c \in \mathbb{C}^\times.$$

In particular, $R_{L:M}(x)$ remains within the same class of Zolotarëv fractions.
(B) Conversely, let $R(x)$ be a degree n twisted Zolotarëv fraction with the set B of four distinct critical values. We exhibit two nonexceptional lattices $M \subset L$ such that $R(x)$ is in the same class as $R_{L:M}(x)$.

The Riemann–Hurwitz formula suggests that there are exactly four noncritical points in the pre-image $R^{-1}B$. We designate this set B_1. Let T (resp. T_1) be a torus doubly covering Riemann sphere with branching at the points of B (resp. B_1). In what follows, we supply all the objects related to the torus T_1 with the subindex 1 by default. When the fraction R changes in its class, the branching sets B and B_1 are subjected to the action of $PSL_2(\mathbb{C})$, so we assume without loss of generality that $\infty \notin B, \; B_1$. The affine part of the elliptic curve T is now given by the equation

$$(x, w) \in \mathbb{C}^2 : \qquad w^2 = \prod_{e \in B} (x - e), \tag{4}$$

and the curve admits an involution $J(x, w) := (x, -w)$ with four fixed points that project to the set B. We claim that there exists an isogeny $\tilde{R} : \; T_1 \to T$ of the tori that commutes with the involutions J, J_1, and its descent to the bases T_1/J_1 and T/J coincides with $R(x)$. The isogeny is a (essentially unique) rational map

$$\tilde{R} : T_1 \ni (x_1, w_1) \quad \to \quad (R(x_1), w_1 S(x_1)) := (x, w) \in T,$$

where $S(\cdot)$ is a rational function with simple zeros at the critical points of $R(\cdot)$, double zero at infinity, and double poles at the poles of $R(\cdot)$.

Let us recall some facts about the universal covering $\tilde{T}$ of the torus. The latter is the space of paths on T starting at the marked point modulo their homotopy. To introduce a global complex coordinate on the universal covering, we choose a (unique up to nonzero factor) holomorphic differential on the torus, for instance, $\mathrm{d}\eta = w^{-1} dx$ in the algebraic model (4) of the torus. Now we assign a complex number $u[\gamma] := \int_\gamma \mathrm{d}\eta$ to a path γ. The torus itself is a factor of the complex plane by the group of cover transformations which is realized as translations by the elements of the lattice

$$L := \int_C \mathrm{d}\eta, \qquad C \in H_1(T, \mathbb{Z}).$$

If the marked point on the torus is fixed by the involution, then the lift of J to $\tilde{T}$ is just a central symmetry:

$$u[J\gamma] = \int_{J\gamma} \mathrm{d}\eta = \int_\gamma J^* \mathrm{d}\eta = -\int_\gamma \mathrm{d}\eta = -u[\gamma].$$

The natural projection $\mathbb{C} \to \mathbb{C}P_1 = T/J = \mathbb{C}/L^+$ is a superposition of a linear fractional map (which we do not specify here) and the function $x_L(u) := \wp(u|L)$.

The torus T and its covering torus T_1 share the universal covering in a natural way: back and forth mappings $\tilde{T} \leftrightarrow \tilde{T}_1$ are given by the projection and lifting of paths, provided the marked points on the tori agree. For the identified points of $\tilde{T}$ and $\tilde{T}_1$ to have the same complex coordinate, we take the pullback of the holomorphic differential $d\eta$ as a distinguished differential $d\eta_1$ on T_1. Now the identified paths $\gamma_1 \subset T_1$ and $\tilde{R}\gamma_1 := \gamma \subset T$ will share the coordinate

$$u_1[\gamma_1] := \int_{\gamma_1} d\eta_1 = \int_{\gamma_1} \tilde{R}^* d\eta = \int_{\tilde{R}\gamma_1} d\eta := u[\gamma].$$

With this choice of the global coordinate on $\tilde{T}_1$, the covering torus will be a factor of the complex plane by the sublattice of L:

$$M = L_1 := \int_{H_1(T_1,\mathbb{Z})} d\eta_1 = \int_{H_1(T_1,\mathbb{Z})} \tilde{R}^* d\eta = \int_{\tilde{R}H_1(T_1,\mathbb{Z})} d\eta \subset \int_{H_1(T,\mathbb{Z})} d\eta := L.$$

The consistent choice of the marked points on the tori, $(e_1, 0) \in T_1$, $e_1 \in B_1$, and respectively $(e, 0) \in T$, $e := R(e_1) \in B$, leads to a commutative square:

$$\begin{array}{ccc} \tilde{T}_1 = \mathbb{C} & \stackrel{id}{\longrightarrow} & \mathbb{C} = \tilde{T} \\ l_1 \circ x_M(u) \downarrow & & \downarrow l \circ x_L(u) \\ \mathbb{C}/M^+ = \mathbb{C}P^1 & \stackrel{R(x)}{\longrightarrow} & \mathbb{C}P^1 = \mathbb{C}/L^+, \end{array}$$

with linear fractional functions $l_1,\ l$, where from the representation (3), it follows that

$$R = l \circ x_L \circ x_M^{-1} \circ l_1^{-1}.$$

The arising pair of lattices $M \subset L$ are not exceptional since they generate a function $R_{L:M}(x)$ with four critical values, as many as $R(x)$ has. Another choice of the distinguished differential $d\eta$ on T leads to rescaling/rotation of the lattices. This ends the proof. □

3 Zolotarëv and Chebyshev–Blaschke are Classmates

Classical Zolotarëv fractions and Chebyshev–Blaschke products as follow from Examples (A) and (B) of Sect. 2 are determined by proportional pairs of embedded lattices $(L(\tau), L(n\tau)) = n\tau\ (L_n(\tau_1), L_1(\tau_1))$ with $\tau_1 = -4/(n\tau)$. By Theorem 1, two mentioned rational functions differ by normalizations of dependent and independent variables only. We show this in a straightforward way below.

First we establish the key identity

$$cd(2K(\tau)u|\tau) = l_\tau \circ cd(iK'(\tau_0)u|\tau_0), \tag{5}$$

where $\tau_0 = -4/\tau$; $K(\tau)$, $K'(\tau)$ are complete elliptic integrals of modulus τ (they may be defined in terms of theta constants with the named modulus [2,11]) and l_τ is a linear-fractional map:

$$l_\tau(w) = -\frac{1}{\sqrt{k}}\frac{\sqrt{k_0}w-1}{\sqrt{k_0}w+1}, \qquad \frac{1}{\sqrt{k}} = \frac{1+\sqrt{k_0}}{1-\sqrt{k_0}}, \tag{6}$$

where the square root of the Jacobi's modulus $\sqrt{k}(\tau) = \theta_2/\theta_3$ is holomorphic in the upper half-plane and $\sqrt{k_0} := \sqrt{k}(\tau_0)$.

To prove (5), we note that $cd(\dots)$ functions on both sides of the equality are even, degree 2 with the common period lattice $Span_{\mathbb{Z}}(2,\tau)$. Hence they differ by a linear fractional map that may be reconstructed if we evaluate it at three points. Taking $u = \frac{1}{2},\ \frac{1}{2}+\frac{\tau}{2},\ 0$, we see that l_τ sends three points $(k_0)^{-1/2}, -(k_0)^{-1/2}, 1$ to respectively $0,\ \infty,\ 1$, wherefrom the expression (6) for the linear fractional map follows.

Let us now make a transformation of Chebyshev–Blashke product:

$$y_{n,\tau}(u) := \sqrt{k(n\tau)}cd(2K(n\tau)nu|n\tau) = \sqrt{k(n\tau)}l_{n\tau}\circ cd(iK'(\tau_1)nu|\tau_1) =$$

(keep in mind the notations: $\tau_1 := -4/(n\tau)$; $l_\tau^0 := \sqrt{k(\tau)}l_\tau$ and the identity [2] $cd(u|\tau) = sn(u+K(\tau)|\tau)$)

$$= l_{n\tau}^0\circ cd(4K(\tau_1)u/\tau|\tau_1) = l_{n\tau}^0\circ sn(K(\tau_1)(4u/\tau+1)|\tau_1) = l_{n\tau}^0\circ x_{\tau_1}(4u/\tau+1).$$

For $n=1$ we have in particular

$$y_{1,\tau}(u) := l_\tau^0\circ x_{\tau_0}(4u/\tau+1),$$

with $\tau_0 = n\tau_1 = -4/\tau$ defined in the above passage. Introducing $1+4u/\tau$ as a new independent variable in the plane, we see that the Chebyshev–Blaschke product is just a renormalization of the Zolotarëv fraction:

$$Y_n(\cdot|\tau) := y_{n,\tau}\circ y_{1,\tau}^{-1} = l_{n\tau}^0\circ x_{\tau_1}\circ x_{n\tau_1}^{-1}\circ (l_\tau^0)^{-1} = l_{n\tau}^0\circ Z_n(\cdot|\tau_1)\circ (l_\tau^0)^{-1}.$$

Now it would be useful to obtain the numerous properties [9] of Y_n, just transferring them from the properties of Zolotarëv fractions.

4 Accounting of the (Sub)lattices

We start with enumeration of given index n sublattices of a rank 2 lattice. Fix any 'standard' basis in L; a basis in the sublattice M is related to the former by an integer 2×2 matrix Q, $|det Q| = |L:M| = n$. The change of a basis in the sublattice results in multiplying Q on the left by an integer matrix with determinant ± 1. Thus all matrices Q are collected into classes of equivalence representing the same sublattice M. It is easy to check that each class contains a unique matrix of the kind

$$Q = \left\|\begin{matrix} p & l \\ 0 & q \end{matrix}\right\|, \qquad p,q>0; \quad q>l\geq 0; \quad pq=n,$$

known as a Hermitian normal form of integer matrix [8].

Remark The representation of a sublattice M by a matrix as above depends on the choice of the 'standard' basis in the ambient lattice L. In other words, there is a right action of invertible integer matrices on Hermitian normal forms.

Each positive integer divisor q of index n gives us q sublattices that differ by the value of an off-diagonal element of matrix Q; therefore we arrive at:

Lemma 1 *A rank two lattice has $\sigma_1(n)$ sublattices of a given index n, where $\sigma_1(n)$ is the sum of all positive integer divisors of n.*

Example $\sigma_1(11) = 1 + 11 = 12$; $\sigma_1(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.

The above Theorem 1 reduces counting classes of degree n twisted Zolotarëv fractions to the enumeration of index n sublattices of a given lattice modulo dilations and rotations.

4.1 General Lattice

When $j(B) \neq 0, 1$, the lattice L has no symmetries: $L = \mu L$ means that $\mu = \mp 1$. Each index n sublattice M of L gives a class of degree n Zolotarëv fractions, whose number is therefore $\sigma_1(n)$. In case $n = 4$, the answer is one less, as we throw away the exceptional sublattice $M = 2L$.

4.2 Square Lattice

When $j(B) = 1$, the corresponding lattice L is equivalent to Gaussian integers $\mathbb{Z}[i]$. It survives after multiplication by i; therefore the sublattice $M \subset L$ will be in the same class as iM and will be counted twice unless $M = iM$. Each square (sub)lattice M is uniquely determined by its nonzero element $e = p + iq$ of minimal length lying in the sector $\{0 \leq Arg(u) < \pi/2\}$. This sublattice has index

$$(|e|^2) := n = p^2 + q^2, \qquad p > 0;\ q \geq 0. \tag{7}$$

Given n, the number of ordered integer solutions p, q of the above equation (7) is known as $S_2(n)$; for instance, $S_2(25) = 3$ since $25 = 5^2 + 0^2 = 4^2 + 3^2 = 3^2 + 4^2$. In this notation, the number of classes of degree n twisted Zolotarëv fractions with $j = 1$ is $\frac{1}{2}(\sigma_1(n) + S_2(n))$.

4.3 Hexagonal Lattice

When $j(B) = 0$, the corresponding lattice L is equivalent to $\mathbb{Z}[\epsilon]$, $\epsilon = \exp(i\pi/3)$. It survives after the rotation by $\pi/3$. The sublattice M will be in the same class as ϵM and $\epsilon^2 M$ and will be counted thrice unless the sublattice M is hexagonal itself. Each hexagonal (sub)lattice M is uniquely determined by its nonzero element $e = p + \epsilon q$

Table 1 Number of classes of twisted Zolotarëv fractions, given the j-invariant and the degree n

	$j=0$	$j=1$	$j\neq 0,1$
$n=4$	2	3	6
$4\neq n>2$	$\frac{1}{3}(\sigma_1(n)+2h(n))$	$\frac{1}{2}(\sigma_1(n)+S_2(n))$	$\sigma_1(n)$

of minimal length lying in the sector $\{0\leq Arg(u)<\pi/3\}$. This sublattice has an index

$$(|e|^2):=n=p^2+pq+q^2,\qquad p>0;\ q\geq 0. \tag{8}$$

Given n, we designate by $h(n)$ the number of ordered integer solutions p,q of equation (8); for instance, $h(13)=2$ since $13=1^2+1\cdot 3+3^2=3^2+3\cdot 1+1$. In this notation, the number of classes of degree n Zolotarëv fractions with $j=0$ is $\frac{1}{3}(\sigma_1(n)+2h(n))$.

The results of this section are summarized in the Table 1.

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