Mosaic Ranks for the Inverses to Band Matrices

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ABSTRACT

After reminding the definition of mosaic ranks, we estimate them from above for the inverses to band matrices. The estimate grows logarithmically with the matrix size. The result presented in this note should be compared with the well-known descriptions of the inverses to band matrices as semiseparable matrices. Our approach, still, excells in that it holds under much weaker assumptions.

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1 Introduction

To begin with, we remind some definitions and well-known facts.

Let $1 \le p, q \le n$. We say that $A = [a_{ij}]_{ij=1}^{n}$ is a (p, q) band matrix if $a_{ij} = 0$ whenever $i - j < -p$ or $i - j > q$.

It is very important what comes along the *extreme* diagonals $i - j = -p$ and $i - j = q$. If there is a nonzero entry on each of them, then p and q are correctly defined upper and lower bandwidth. If all their entries are nonzeroes, then A is called a *strict* (p, q) band matrix [10].

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Denote by $\mathcal{U}_p$ and $\mathcal{L}_q$ the spaces of upper and lower triangular matrices of order p and q, respectively. We say that A is a (q,p) semiseparable matrix if, first,

$$A = S + \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}, \quad \operatorname{rank} S = q, \quad U \in \mathcal{U}_{n-q}, \tag{1}$$

and, second,

$$A = R + \begin{bmatrix} 0 & 0 \\ L & 0 \end{bmatrix}, \quad \operatorname{rank} R = p, \quad L \in \mathcal{L}_{n-p}. \tag{2}$$

It is known for years that the inverses to band matrices still capture rather imposing structure. We all are aware of the following beautiful, well-known, and rediscovered several times theorem:

A nonsingular matrix is a strict (p,q) band matrix if and only if its inverse is a (q,p) semiseparable matrix.

The proof can be found, for example, in [10] (see also [1, 2, 7, 15]). However, in contrast to the definition and proof given in [10], we need not to assume additionally that the diagonals $i-j=-q$ in S and R (and $i-j=p$ also) are entry-wise different. The above theorem can be proved without this.

In fact, the proof [2] can be very short and clear. Suppose, first, that A is a nonsingular (q,p) semiseparable matrix. Rewrite (1) in the block form $A = \begin{bmatrix} u_1v_1 & u_1v_2+U \\ u_2v_1 & u_2v_2 \end{bmatrix}$. We conclude immediately that the blocks u_2 and v_1 are nonsingular. The Schur complement to the (2,1) block is $u_1v_2 + U - (u_1v_1)(u_2v_1)^{-1}(u_2v_2) = U$. It implies that U is nonsingular and $A^{-1} = \begin{bmatrix} * & * \\ U^{-1} & * \end{bmatrix}$. Using the same arguments, from (2) we derive that $A^{-1} = \begin{bmatrix} * & L^{-1} \\ * & * \end{bmatrix}$, which completes the proof of the "only if" part.

[2] The idea goes back probably to D.K.Faddeev.

Concerning the "if" part, set $A = \begin{bmatrix} a & b \\ U & c \end{bmatrix}$ and remember the Frobenius formulas: with $h = (b - aU^{-1}c)^{-1}$ it holds that

$$A^{-1} = \begin{bmatrix} U^{-1}ch & U^{-1} - U^{-1}chaU^{-1} \\ h & -haU^{-1} \end{bmatrix}.$$

Still, one ought to recover from this that

$$A^{-1} = \begin{bmatrix} U^{-1}ch \\ h \end{bmatrix} \begin{bmatrix} I & aU^{-1} \end{bmatrix} + \begin{bmatrix} 0 & U^{-1} \\ 0 & 0 \end{bmatrix}.$$

Thus, we have got (1), and the same, just "transposed", logic leads to (2).

The above proof is by no chance a purpose of this paper. Getting on to the purpose, we note that the theorem under discussion stands only for *strict* band matrices. It is no longer valid when a zero occurs on the extreme diagonals. For tridiagonal matrices, however, such zeroes make things just simpler (reducing the case to a block diagonal matrix with strict tridiagonal blocks). All the same, all is not that simple for arbitrary band matrices.

The goal of this paper is probably two-fold. First, we discuss what kind of structure is maintained by the inverses for arbitrary, not necessarily strict, band matrices. To this end, we introduce a concept of (p, q) weakly semiseparable matrices and prove that their inverses remain in the same class (Section 2). It is pertinent to note that any (p, q) band matrix is also a (p, q) weakly semiseparable matrix.

Second, we want to say about the relation between semiseparable matrices and the mosaic ranks introduced of late by the author [11, 12, 13]. That concept is simple and useful for the analysis of structure of matrices arisen as discrete analogs of integral operators [13, 5]. More or less indirectly, it is related to the constructions considered in [3, 4, 6, 8, 9, 14]. In Section 3 we remind the notion of mosaic ranks. Then we estimate the mosaic ranks for weakly semiseparable matrices.

2 Weakly semiseparable matrices

Assume that an $n \times n$ matrix A is such that the rank of any submatrix located in its upper triangular (and lower triangular) part is at most p (and q). Then A is said to be a (p, q) *weakly semiseparable* matrix.

Theorem 2.1 *A nonsingular matrix is (p, q) weakly semiseparable if and only if its inverse possesses the same property.*

Proof. It is sufficient to prove the following assertion. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where the diagonal blocks are square. Then $\operatorname{rank} B_{12} = \operatorname{rank} A_{12}$.

If A_{11} is nonsingular, then this emanates from the Frobenius formulas. Otherwise, consider matrices $A(t) = A + tI$ with a real parameter t. For all sufficiently small $0 < t < \varepsilon$ the block $A_{11}(t)$ is nonsingular (it follows that $A_{22}(t)$ is also nonsingular). Consequently,

$$\operatorname{rank} B_{12}(t) \leq \operatorname{rank} A_{12}(t) \quad \forall\, 0 < t < \varepsilon.$$

Obviously, $A(t) \to A$ as $t \to 0$, and, in the limit, we come to

$$\operatorname{rank} B_{12} \leq \operatorname{rank} A_{12}.$$

(It might be worthy to remark that we can not pass to the limit in the Frobenius formulas.)

There is only a tiny little step from the above assertion to the complete proof of the theorem. □

For band matrices, we easily obtain the following corollary.

Corollary. *The inverse to a nonsinular (p, q) band matrix is a (p, q) weakly semiseparable matrix.*

3 Mosaic ranks

We now present the concept of mosaic ranks. It is related to some urgent practical calls for efficient algorithms for large dense matrices. In our approach, we try viewing large dense matrices as specifically structured (more typical for applications, we look for structured approximations).

A basic kind of structure we choose are so-called *skeletons*. We use this name for matrices of the form uv^T, where u and v are column vectors. Using the skeleton expansion $A = \sum_{i=1}^{r} u_i v_i^T$ we can calculate $y = Ax$ in only $2rn$ coupled operations (instead of n^2) using only $2rn$ memory locations (instead of n^2).

In the general case, when A is $m \times n$, the compressed memory is defined by the formula

$$\text{MEMORY} = \text{RANK} \cdot (m+n).$$

For arbitrary matrices (prospectively nonsingular), the mosaic rank is introduced so that the formula

$$\text{MEMORY} = \text{MOSAIC RANK} \cdot (m+n)$$

be still valid [12, 13].

If B is a submatrix for a $m \times n$ matrix A then $\Gamma(B)$ denotes the $m \times n$ matrix with the same block B and zeroes elsewhere. A system of blocks A_i is called a *covering* of A if

$$A = \sum_i \Gamma(A_i),$$

and a *mosaic partitioning* of A if the blocks have no common elements.

For any given covering, there might be far too few skeletons in every block (the fewer the better). The *mosaic rank* of A is defined as

$$\operatorname{mr} A = \sum_i \operatorname{mem} A_i \;\; / \;\; (m+n), \tag{3}$$

where

$$\operatorname{mem} A_i = \min \{ m_i n_i \ , \ \operatorname{rank} A_i (m_i + n_i) \}. \tag{4}$$

Mosaic ranks for different mosaic partitionings of the same matrix may differ. Of course, we can always consider the optimal mosaic rank as the minimum over all mosaic partitionings. However, any time we write $r = \operatorname{mr} A$

below it means only that there exists a mosaic partitioning whose mosaic rank is equal to r.

Below we rely on the following lemma proposed in [12, 13].

Lemma 3.1 *Given a matrix A of the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m_1 \times n_1},\ A_{22} \in \mathbb{C}^{m_2 \times n_2}.$$

assume that

$$|(m_1 + n_1) - (m_2 + n_2)| \le q(m+n), \quad q \ < \ 1, \tag{5}$$

and there exist mosaic partitionings such that

$$\mathrm{mr}\, A_{ii} \le c \log_2^{k+1}(m_i + n_i),\ i = 1, 2; \quad \mathrm{mr}\, A_{ij} \le r \log_2^{k}(m_i + n_j),\ i \ne j, \tag{6}$$

for some $k \ge 0$, and, moreover,

$$c \ge \frac{r}{\log_2 \frac{2}{q^2+1}}.$$

Then for A, there exists a mosaic partitioning such that

$$\mathrm{mr}\, A \le c \log_2^{k+1}(m+n). \tag{7}$$

We are now in a position to formulate the following result.

Theorem 3.1 *For any nonsingular (p, q) weakly semiseparable matrix A_n of order $n > n_0 > 0$,*

$$\mathrm{mr}\, A_n \le c(n_0)\ \max\{p, q\}\ \log_2 2n,$$

where

$$c(n_0) = \frac{1}{\log_2 \frac{2}{\frac{1}{4n_0^2}+1}}.$$

Proof. If n is a power of two, we subdivide A_n into square blocks of order $n/2$ and, then, proceed by induction:

$$\frac{np + nq + 2n\ \max\{p,q\}\ \log n}{2n} = \max\{p,q\}\ \left(\frac{1}{2} + \log n\right) \leq \max\{p,q\}\ \log 2n.$$

Thus, we obtained the estimate even sharper than what we are after.

However, the case of an arbitrary n seems to be more tricky, and that is where we luckily fall back on Lemma 3.1. For an arbitrary n, we can always have the square diagonal blocks of order n_1 and n_2 with $|n_1 - n_2| \leq 1$. Therefore, we may set $q = 1/2n_0$. Then, we apply Lemma 3.1 straightforward with $k = 1$ and, obviously, $r = \max\{p,q\}$. $\square$

We can interpret the result also as follows: for any $\varepsilon > 0$, it holds

$$\operatorname{mr} A_n \leq (1+\varepsilon)\ \max\{p,q\}\ \log_2 2n$$

for all n sufficiently large.

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