A Mosaic Preconditioner for a Dual Schur Complement*

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1 Introduction

The paper is devoted to designing an interface preconditioner for a Schur complement associated with Lagrange multipliers. After brief overview of the problem in Introduction, in Section 2 we discuss a domain decomposition method with Lagrange multipliers on the interface. In Section 3 we outline the general framework of the solution of saddle-point systems which result from the above domain decomposition. Section 4 is devoted to the construction of the interface preconditioner for the saddle-point Schur complement, which is the main goal of the paper. In Section 5 we review the mosaic approximation to be used in the preconditioner and analyze the mosaic preconditioner. In Section 6 we present numerical experiments illustrating the basic properties of the interface preconditioner.

The main purpose of the domain decomposition is *modularity*. By modularity we imply an ability to use only subdomain solvers while solving global problems. By subdomain solvers one could understand either exact solvers (direct or iterative) or a preconditioner. The latter is more favorable because of its flexibility. From parallelization point of view, the most attractive are non-overlapping domain decompositions. However, these methods require an additional problem dependent constituent, an interface preconditioner. Designing an interface preconditioner is the key ingredient for many domain decomposition techniques. There exists several ways of construction such a preconditioner. One could solve auxiliary boundary value problems in subdomains [16, 14, 4, 5, 7, 8, 9], use an approximate inverse of the interface Schur complement [17, 11, 6], construct spectrally equivalent interface operators [1, 18, 20]. In this paper we want to exploit trace normalizations [18, 19], in order to construct an interface preconditioner. The basic idea is to approximate a dense matrix due to the trace normalization, by a sparse matrix whose action on vector is easy-to-evaluate. In 2D case of the substructuring domain decomposition method, such an attempt was discussed in [21]. Here, we address preconditioning a dual Schur complement, that makes the problem easier even in 3D case. The proposed

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preconditioner may be implemented as a gray box having on input the coefficients of the problem and the interface mesh. The mosaic approximation technique [23] used as a gray box, is associated with the fast multipole methods [27].

2 Domain decomposition with Lagrange multipliers

Let a polyhedral domain $\Omega \subset \mathbf{R}^3$ with a Lipschitz boundary $\partial\Omega$ be decomposed into m non-overlapping regular shaped polyhedral subdomains Ω_i, $1 \le i \le m$, i.e., $\bar{\Omega} = \cup_{i=1}^{m} \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset, 1 \le i \ne j \le m$. We assume this decomposition to be geometrically conforming in the sense that if $\bar{\Theta}_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j \ne \emptyset, i \ne j$, then $\bar{\Theta}_{ij}$ is either a common vertex, a common edge, or a common face of Ω_i and Ω_j. We refer to $\mathcal{S} := \bigcup\{\bar{\Theta}_{ij} : \mid \Theta_{ij} \mid \ne 0, 1 \le i \ne j \le m\}$ as the skeleton of the decomposition. The skeleton is decomposed into faces γ_k: $\mathcal{S} = \bigcup_{k=1}^{K} \bar{\gamma}_k$, where each face γ_k is the entire open face of two adjacent subdomains $\Omega_{M(k)}$ and $\Omega_{\bar{M}(k)}, 1 \le M(k) \ne \bar{M}(k) \le m$, i.e., $\gamma_k = \Theta_{M(k),\bar{M}(k)}$.

Let Ω_i^h be a conformal simplicial regular triangulation of Ω_i, $i = 1, \ldots, m$. We assume that the meshes Ω_i^h match on the skeleton: $\Omega_{M(k)}^h|_{\gamma_k} = \Omega_{\bar{M}(k)}^h|_{\gamma_k}$, $k = 1, \ldots, K$. Therefore, the conformal triangulation of Ω may be defined as the union of Ω_i^h:

$$\Omega^h = \bigcup_{i=1}^{m} \Omega_i^h.$$

We denote by V_i^h and W^h the spaces of P_1 conforming finite elements on triangulations Ω_i^h and Ω^h, respectively.

Let the bilinear form $a(\cdot,\cdot) : V \times V \to \mathbf{R}$, and the functional $l(\cdot) : V \to \mathbf{R}$ are given by

$$a(v,w) := \sum_{i=1}^{m} a_i(v,w), \quad a_i(v,w) := \int_{\Omega_i} [\rho \nabla v \cdot \nabla w + \varepsilon v w] dx, \quad l(v) := \sum_{i=1}^{m} \int_{\Omega_i} f v dx, \ f \in L_2(\Omega).$$

For simplicity we assume that $\varepsilon(x) = \varepsilon_i \equiv const_i > 0$, $\rho(x) = \rho_i \equiv const_i > 0$ in Ω_i, $i = 1, \ldots, m$. The finite element problem is

Find $w \in W^h$ such that

$$a(w,v) = l(v), \quad v \in W^h. \tag{1}$$

In order to decompose the problem (1), we introduce a pair of discrete spaces:

$$V^h := \prod_{i=1}^{m} V_i^h \quad \text{and} \quad \Lambda^h := \prod_{k=1}^{K} \Lambda^h(\gamma_k),$$

where

$$\Lambda^h(\gamma_k) \ := \ \Big\{ v = \sum_{i \in \{\mathcal{N}(\gamma_k^h)\}} \beta_i \boldsymbol{\delta}(x_i) \Big\}. \tag{2}$$

Here $\mathcal{N}(\gamma_k^h)$ is the set of all nodes of the mesh γ_k^h on the entire face $\bar{\gamma}_k$. Notation $\boldsymbol{\delta}(x)$ stands for the Dirac function.

The space V^h is composed of decoupled subdomain spaces V_i^h, and the space Λ^h is the space of Lagrange multipliers. The macro-hybrid primal variational formulation is: Find $(u, \lambda) \in V^h \times \Lambda^h$ such that

$$\begin{aligned} a(u,v) + b(\lambda, v) &= l(v), \quad v \in V^h, \\ b(\mu, u) &= 0, \quad \mu \in \Lambda^h, \end{aligned} \tag{3}$$

where

$$b(\mu, v) := \sum_{k=1}^{K} b_k(\mu, v), \quad b_k(\mu, v) :=< \mu, [v]_J >_{\gamma_k},$$

$[v]_J|_{\gamma_k} := v|_{\Omega_{\bar{M}(k)}} - v|_{\Omega_{M(k)}}$, and $< \cdot, \cdot >_{\gamma_k}$ refers to the dual pairing[1] between $H^{-1}(\gamma_k)$ and $H^1(\gamma_k)$. In spite of lack of a continuous closure, the following holds true in the above discrete case [10]:

Lemma 2.1 *Let w and u be the solutions of of (1) and (3), respectively.*

1. *There exists such $\hat{\lambda} \in \Lambda^h$ that the pair $\left\{ \{w|_{\Omega_i}\}_{i=1}^m, \hat{\lambda} \right\}$ is the solution of (3);*
2. *The function $\hat{w} \in W^h$, $\hat{w}|_{\Omega_i} = u_i$, is the solution of (1).*

The equivalence of (1) and (3) allows us to replace (1) by (3). The advantages of such a replacement are discussed below.

3 Block diagonal preconditioner

The finite element problem (3) results in the system of linear algebraic equations in the saddle-point form:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} A_1 & & & 0 & B_1^T \\ & \cdot & & & \cdot \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ 0 & & & A_m & B_m^T \\ B_1 & \cdot & \cdot & B_m & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_m \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_m \\ 0 \end{bmatrix}, \tag{4}$$

where the block representations of the matrices A and B are associated with the definition of the spaces V^h and Λ^h, while the matrices A, B and the vector f are specified by the bilinear form $a(u, v)$, $b(\mu, v)$ and the functional $l(v)$, respectively. Under the assumptions made, matrices A_i are symmetric positive definite. However, the whole matrix of system (4) may be singular, if the matrix B is not a full rank matrix. It is the so-called redundancy case: the number of neighbor-subdomain pairs (Lagrange multiplier degrees of freedom) related to an interface node, may be larger than the minimum number of ties providing continuity of a grid function at this node.

[1] We consider only discrete formulations, for which $v|_{\Omega_{M(k)}} \in H^1(\gamma_k)$, since $v \in V_i^h$.

Let us assume for the moment that the matrix B is a full rank matrix. In order to solve the linear problem (4) one can use the generalized Lanczos method with the preconditioner

$$R = \begin{bmatrix} R_A & 0 \\ 0 & R_\lambda \end{bmatrix}. \tag{5}$$

The symmetric positive definite matrix R_A stands for a preconditioner for the subdomain problems specified by the matrix A, and R_λ stands for a preconditioner for the interface problem associated with the matrix $S = BA^{-1}B^T$.

Lemma 3.1 ([11, 6]) *Let the symmetric positive definite matrices R_A and R_λ be spectrally equivalent to the matrices A and $S = BA^{-1}B^T$, respectively, with positive constants c_1, c_2, c_3, c_4. Then the boundaries of the segments $[d_1, d_2]$, $[d_3, d_4]$, $d_1 \le d_2 < 0 < d_3 \le d_4$, that contain the spectrum of the matrix $R^{-1}\mathcal{A}$, $\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$, depend only on the values of c_1, c_2, c_3, c_4.*

Corollary 3.1 *Under the above assumption on the matrix R the convergence rate of the preconditioned Lanczos method of minimal iterations does not depend on the dimension of the matrix $\mathcal{A}$ if it is applied to the problem (4).*

In the case when the matrix B is not a full rank matrix, the convergence of the iterative methods is understood in a subspace of the original discrete space. The subspace is formed by the Lagrange multipliers from $range(B)$.

In papers [6, 11] the matrix R_A is proposed to be a block diagonal matrix such that each block of R_A serves as a preconditioner for the corresponding diagonal block A_i of matrix A. These blocks are independent of each other and may be chosen taking the advantage of properties of the subdomain meshes, approximations, operators, *etc.* The challenge is the construction of an efficient interface preconditioner R_λ. It should be parallel, fast and robust with respect to bad parameters of the discrete problem and its decomposition (coefficients, meshes, number of subdomains). In the following we focus on developing such an interface preconditioner.

Remark 1. Construction of a preconditioner for the dual Schur complement $S = BA^{-1}B^T$ is the basic issue for efficient iterative solution of saddle-point problems [3]. In particular, instead of the framework of Lemma 3.1, one could simply reduce the problem (4) to its interface Schur complement counterpart, assuming that matrices A_i, $i = 1, \ldots, m$, may be factorized:

$$S\lambda = F, \quad F = BA^{-1}f. \tag{6}$$

Problem (6) is also referred to as the dual Schur complement problem, since the Schur complement here is associated with the Lagrange multipliers rather than the trace of a solution. Although the matrix S is dense, it is never generated as it is. Instead, iterative techniques are used for the solution of (6). Computation of residual requires only multiplication of S by a vector, which may be done easily once the matrices A_i

are factorized. Moreover, the multiplication is easy to parallelize, as well as recovering the solution $u = A^{-1}(f - B^T\lambda)$. The condition number of S (cond$S$) is affected by the number of nodes on the skeleton, the meshes in the subdomains, the coefficients in the bilinear forms, and the number of subdomains. Usually, the condition number of S is large, and the preconditioning is inevitable.

Remark 2. In this work, we consider the case $\varepsilon_i > 0$. It provides non-singularity of matrices A_i, $i = 1, \ldots, m$. On the other hand, it may cause an additional dependence of cond$S$ on ε. The case $\varepsilon_i = 0$ is considered by the Finite Element Tearing and Interconnecting (FETI) method, introduced by Farhat and Roux [4, 5]. Let $A^\dagger$ stand for the pseudo-inverse of A, problem (6) is reformulated as:

$$BA^\dagger B^T\lambda = BA^\dagger f + BR\alpha, \tag{7}$$

where α has to be determined and matrix R spans the nullspace of A: $range(R) = KerA$. If we denote $G := BR$, $P := I - G(G^TG)^{-1}G^T$, problem (7) is equivalent to

$$\left\{ \begin{array}{rcl} BA^\dagger B^T\lambda & = & BA^\dagger f, \\ G^T\lambda & = & R^T f. \end{array} \right. \tag{8}$$

The FETI method is the solution of (8) with a preconditioned projected CG method. In other words, it is the PCG method where all guesses satisfy $G^T(\lambda_k - \lambda_0) = 0$.

4 Interface preconditioner

For simplicity of presentation, we consider three cases. The case with severe topological restrictions will be considered in detail, and an intermediate and a general case will be just outlined.

4.1 Case 1: no crossedges

Crossedges are defined to be the edges forming the skeleton $\mathcal{S}$. Since the faces $\bar{\Theta}_{M(k),\bar{M}(k)}$, $k = 1, \ldots, m$, do not intersect each other, we have as many degrees of freedom for Lagrange multiplier space, as the number of ties to be imposed in order to get continuity of the solution across the interfaces. It implies the non-redundancy and the full rank of the matrix B. A very important feature of such a decomposition is that a normalization of the discrete trace space on the skeleton may be derived explicitly [19], in construct to a general case. Due to the topological restrictions and local nature of the matrix B, the matrix $S = BA^{-1}B^T$ is spectrally equivalent to a block diagonal matrix $blockdiag\{S_k\}$ [], each block S_k being associated with an entire face $\bar{\Theta}_{M(k),\bar{M}(k)}$, $k = 1, \ldots, K$:

$$S \sim blockdiag\{S_k\}. \tag{9}$$

Moreover, nonzero entries of matrices B_i, $i = 1, \ldots, m$, are either 1 or -1, according to (2). Hence,

$$S_k = F_{M(k),k}A^{-1}_{M(k),k}F^T_{M(k),k} + F_{\bar{M}(k),k}A^{-1}_{\bar{M}(k),k}F^T_{\bar{M}(k),k},$$

where matrix $F_{M(k),k}$ with entries 1 or 0 map the degrees of freedom in subdomain $\Omega_{M(k)}$ into Lagrangian degree of freedom $\mathcal{N}(\gamma_k^h)$. Let we split the nodes in $\Omega^h_{M(k)}$ into two groups, the first is formed by the nodes lying on $\bar{\Theta}_{M(k),\bar{M}(k)}$ and the second is formed by the remained nodes. Then the block representation of matrix $A_{M(k)}$ is

$$A_{M(k)} = \begin{bmatrix} A^k_{M(k)} & A^{kI}_{M(k)} \\ A^{Ik}_{M(k)} & A^{I}_{M(k)} \end{bmatrix}, \tag{10}$$

and we readily get

$$S_k = Q^{-1}_{M(k),k} + Q^{-1}_{\bar{M}(k),k}, \tag{11}$$

$$Q^{-1}_{M(k),k} = A^k_{M(k)} + A^{kI}_{M(k)} \left(A^I_{M(k)}\right)^{-1} A^{Ik}_{M(k)},$$

$$Q^{-1}_{\bar{M}(k),k} = A^k_{\bar{M}(k)} + A^{kI}_{\bar{M}(k)} \left(A^I_{\bar{M}(k)}\right)^{-1} A^{Ik}_{\bar{M}(k)}.$$

Let n_k be the number of nodes in $\mathcal{N}(\gamma_k^h)$, u_i be the entry of vector $\vec{u}$ associated with a mesh node $x_i \in \mathcal{N}(\gamma_k^h)$, h_i be the mesh size at x_i and matrix $N_{M(k),k} \in \mathbf{R}^{n_k \times n_k}$ be defined as

$$(N_{M(k),k}\vec{u},\vec{v}) = \sum_{x_i \in \mathcal{N}(\gamma_k^h)} \alpha_i(\rho_{M(k)}, \varepsilon_{M(k)}) \sum_{x_i \neq x_j \in \mathcal{N}(\gamma_k^h)} \frac{(u_i - u_j)(v_i - v_j)}{|x_i - x_j|^3} h_i^2 h_j^2$$

$$+ \sum_{x_i \in \mathcal{N}(\gamma_k^h)} \beta_i(\rho_{M(k)}, \varepsilon_{M(k)}) u_i v_i h_i^2, \quad \forall \vec{u}, \vec{v} \in \mathbf{R}^{n_k}, \tag{12}$$

where

$$\alpha_i(\rho,\varepsilon) = \begin{cases} 0, & \rho/\varepsilon < h_i^2 \\ \rho, & \rho/\varepsilon \geq h_i^2 \end{cases}$$

$$\beta_i(\rho,\varepsilon) = \begin{cases} h_i\varepsilon, & \rho/\varepsilon < h_i^2 \\ (\rho\varepsilon)^{1/2}, & h_i^2 \leq \rho/\varepsilon \leq 1 \\ \varepsilon, & \rho/\varepsilon > 1. \end{cases}$$

Theorem 4.1 ([19])

$$N_{M(k),k} \sim Q_{M(k),k}. \tag{13}$$

The spectral equivalence takes place with constants independent of n_k,ρ, ε.

Now, we assume that

$$\frac{\varepsilon_1}{\rho_1} \sim \frac{\varepsilon_2}{\rho_2} \sim \ldots \sim \frac{\varepsilon_m}{\rho_m}, \tag{14}$$

and we have matrices L_k, $k = 1, \ldots, K$, such that

$$L_k \sim N_{\hat{M}(k),k}, \quad \hat{M}(k) = \begin{cases} M(k), & \rho_{M(k)} \leq \rho_{\bar{M}(k)}, \\ \bar{M}(k), & \rho_{M(k)} > \rho_{\bar{M}(k)}, \end{cases} \tag{15}$$

with constants independent of ε, ρ. Straightforward consequence of (9), (11), (13), (14), (15) is the following

Theorem 4.2 *Let (14) holds true and*

$$L = blockdiag\{L_k\}$$

and

$$c_1 L_k \le N_{\hat{M}(k),k} \le c_2 L_k, \quad k = 1, \dots, K. \tag{16}$$

Then

$$c_3 L^{-1} \le S \le c_4 L^{-1}, \tag{17}$$

where constants c_3, c_4 depend only on c_1, c_2.

According to (17), the preconditioning implies here just a vector multiplication by matrix L. Hence, if the multiplication may be performed efficiently, the goal of construction of an interface preconditioner is achieved. In Section 5 the construction of matrices L_k and verification of (16) will be considered in detail.

4.2 Case 2: no internal crossedges

As in the previous case the topological assumption implies the non-redundancy and full rank of the matrix B. The dual Schur complement is not spectrally equivalent to $blockdiag\{S_k\}$ where blocks S_k are associated with $\bar{\Theta}_{M(k),\bar{M}(k)}$. However, we may ignore the nondiagonal blocks of S and consider the above block diagonal preconditioner $L = blockdiag\{L_k\}$, where L_k satisfy (16). In this case the constants in (17) are not dependent on c_1, c_2 only. They depend on the mesh and the coefficients as well. The dependence on the mesh has a polylogarithmic character [1]. Indeed, one of the basic properties of the matrix S is that, up to a logarithmic factor, $blockdiag\{S\}^{-1} \sim blockdiag\{S^{-1}\}$ ([1],Proposition 2 and [18]). Coupled with (16), it results in polylogarithmic dependence of c_4/c_3. The dependence on the coefficients may be derived as follows. If the entries of nondiagonal blocks of S may be estimated by those of $N_{\hat{M}(k),k}$, $k = 1, \dots, m$, independently of the coefficients, then c_3, c_4 do not depend on the coefficients. For instance (but not necessarily), this is the case when there is a path through faces of neighboring subdomains such that the coefficients are non-increasing or nondecreasing along the path. Otherwise, the nondiagonal blocks are essential and may not be neglected in such a fashion.

4.3 Case 3: general case

In general case normalization of the skeleton trace space is unknown and the matrix B is not a full rank matrix. The remedy is a combination of techniques responsible for different aspects of preconditioning. At the first step, we get rid of the dependence on ε_i and number of subdomains m.

Let $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$, n_{Γ_i} be the number of nodes of $\Gamma_i^h := \partial\Omega_i^h \cap \Gamma_i$, $M_{\Gamma_i} \in \mathbf{R}^{n_{\Gamma_i} \times n_{\Gamma_i}}$ be the boundary mass matrix, d_i be the diameter of Ω_i, $i = 1, \dots, m$.

We introduce the matrix $P_{\Gamma_i} = w_{1,\Gamma_i} w_{1,\Gamma_i}^T$, where $w_{1,\Gamma_i} = \frac{1}{\sqrt{|\Gamma_i|}} e_{\Gamma_i}$, $e_{\Gamma_i} = [1 \dots 1]^T \in \mathbf{R}^{n_{\Gamma_i}}$. We note that $(M_{\Gamma_i} w_{1,\Gamma_i}, w_{1,\Gamma_i}) = 1$, and $P_{\Gamma_i} M_{\Gamma_i}$ are the M_{Γ_i}-orthogonal projectors, $i = 1, \dots, m$. Let $\varepsilon_i \le c\rho_i/d_i^2$ and let $\bar{A}_i$ be a matrix generated on Ω_i^h by the bilinear

form $a_i(u,v)$ with $\varepsilon = \rho_i/d_i^2$. The matrices A_i and $\bar{A}_i$ have the block forms

$$A_i = \begin{bmatrix} A_{\Gamma_i} & A_{\Gamma_i I_i} \\ A_{I_i \Gamma_i} & A_{I_i} \end{bmatrix} \text{ and } \bar{A}_i = \begin{bmatrix} \bar{A}_{\Gamma_i} & \bar{A}_{\Gamma_i I_i} \\ \bar{A}_{I_i \Gamma_i} & \bar{A}_{I_i} \end{bmatrix},$$

where $A_{\Gamma_i}, \bar{A}_{\Gamma_i} \in \mathbf{R}^{n_{\Gamma_i} \times n_{\Gamma_i}}$.

Lemma 4.1 [6, 11] *Under the assumptions made*

$$\left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i} \bar{A}_{I_i}^{-1} \bar{A}_{I_i \Gamma_i}\right)^{-1} + \frac{1}{\varepsilon_i d_i} P_{\Gamma_i} \sim \left(A_{\Gamma_i} - A_{\Gamma_i I_i} A_{I_i}^{-1} A_{I_i \Gamma_i}\right)^{-1}. \tag{18}$$

The spectral equivalence takes place with constants independent of ρ_i, ε_i, d_i.

The above Lemma is used for the construction of a preconditioner to $BA^{-1}B^T$, since $B_i A_i^{-1} B_i^T = B_{\Gamma_i}(A_{\Gamma_i} - A_{\Gamma_i I_i} A_{I_i}^{-1} A_{I_i \Gamma_i})^{-1} B_{\Gamma_i}^T$, where matrix B_{Γ_i} is the interface subblock of B_i, $B_i = (B_{\Gamma_i}, O)$. Using (18) we have

$$BA^{-1}B^T = \sum_{i=1}^{m} B_i A_i^{-1} B_i^T \sim \sum_{i=1}^{m} \frac{1}{\varepsilon_i d_i} B_{\Gamma_i} P_{\Gamma_i} B_{\Gamma_i}^T + \bar{G}, \tag{19}$$

$$\bar{G} = \sum_{i=1}^{m} B_{\Gamma_i} \left(\bar{A}_{\Gamma_i} - \bar{A}_{\Gamma_i I_i} \bar{A}_{I_i}^{-1} \bar{A}_{I_i \Gamma_i}\right)^{-1} B_{\Gamma_i}^T. \tag{20}$$

Of course, since the matrix B may be not a full rank matrix, (19) holds true for vectors in $range(B)$.

Theorem 4.3 [12, 13] *Let B be a full rank matrix and let the symmetric positive definite matrix D be such that the spectrum of $D\bar{G}$ belongs to the interval $[c_1, c_2]$, $0 < c_1 < c_2$ and let*

$$R_\lambda := \sum_{i=1}^{m} \frac{1}{\varepsilon_i d_i} B_{\Gamma_i} P_{\Gamma_i} B_{\Gamma_i}^T + D^{-1}. \tag{21}$$

Then

$$R_\lambda \sim BA^{-1}B^T. \tag{22}$$

The spectral equivalence takes place with constants independent of ρ_i, ε_i, d_i, m and dependent on c_1, c_2.

Matrix R_λ is a modification of D^{-1} by a low rank matrix $XX^T = \sum\limits_{i=1}^{m} \frac{1}{\varepsilon_i d_i} B_{\Gamma_i} P_{\Gamma_i} B_{\Gamma_i}^T$ with $X = \left(\ldots, \frac{1}{\sqrt{\varepsilon_i d_i |\Gamma_i|}} B_{\Gamma_i} e_{\Gamma_i}, \ldots\right)$. The solution of a system with matrix R_λ may be found by evaluations of matrix D:

$$R_\lambda^{-1} = D - DX(I_m^{-1} + X^T D X)^{-1} X^T D,$$

where $I_m \in \mathbf{R}^{m \times m}$ is the identity matrix. Thus, in order to construct a good preconditioner for $BA^{-1}B^T$ we have to find a preconditioner D to $\bar{G}$ such that $D\bar{G} \sim I$ and D is easily multiplied by a vector.

If B is not a full rank matrix, both $\bar{G}$ and D are to be singular but positive definite in $range(B)$. If we assume that D is invariant in $range(B)$, (21), (22) are understood in the subspace $range(B)$.

Thus, we reduced the problem of construction of an interface preconditioner for the dual Schur complement S to preconditioning matrix $\bar{G}$.

At the second step, we can apply the Dirichlet preconditioner for the FETI method [8]. The basic idea is to replace the Schur complement associated with Dirichlet problem by a matrix generating a trace norm on Γ_i. Assumption $\varepsilon_i \leq c\rho_i/d_i^2$ allows us to carry on the basic analysis [8] for our case. However, there are still open algorithmic questions within the above framework. They are to be discussed elsewhere.

5 Mosaic preconditioner

The mosaic-skeleton method [23, 24, 25] constructs a specific ("mosaic") partitioning of a matrix in which most of blocks admit accurate low-rank approximation. For the matrices coming from integral equations, these blocks correspond to a region where the kernel is smooth enough and can be approximated by a degenerate kernel (a short sum of functional skeletons). The method is an algebraic view of several fast approximate multiplication approaches of the last decade [**?**, 27].

Assume that the mosaic-skeleton method is applied to matrices of the form

$$A_n = [f(x_{in}, y_{jn})], \quad 1 \leq i, j \leq n,$$

where $f(x, y)$ is a function of x and y from a bounded region S in the μ-dimensional space, and x_{in} and y_{jn} are the nodes of some meshes.

First, we assume that f is *asymptotically smooth*. It means [24] that there exist $c, d > 0$ and a real number g such that, for $x \neq y$ and all integer $p \geq 0$,

$$|\partial^p f(x, y)| \ \leq \ c_p \, |x - y|^{g-p} \tag{23}$$

where

$$c_p \leq c \, d^p \, p! \, . \tag{24}$$

Here, ∂^p is any p-order derivative in $y = (y_1, \ldots, y_\mu)$:

$$\partial^p = \left(\frac{\partial}{\partial y_1}\right)^{i_1} \cdots \left(\frac{\partial}{\partial y_\mu}\right)^{i_\mu}, \quad i_1 + \ldots + i_\mu = p.$$

Second, we assume the meshes are subject to the following resriction:

$$\tau_n(S') \leq c \, \frac{\text{mes}\, S'}{\text{mes}\, S} \, n,$$

where S' is any subregion of S and τ_n counts how many nodes of the mesh with n nodes fall into S', and mes is the Riemann measure (thus, S and S' are assumed measurable; we could confine ourselves to those S' which are intersections of a finitely many cubes with S); $c > 0$ does not depend on S' and n.

The next theorem is a reformulation of the results from [24, 26].

Theorem. *Under the above two assumptions, for any $\delta > 0$ there are splittings $A_n = T_n + R_n$ where*

$$\mathrm{mem}\,(U_n V_n) = O(n \log^{\mu+1} n), \quad ||R_n||_F = O(n^{-\delta}), \tag{25}$$

and mem *is the lowest possible number of nonzero entries in any pair of matrices U and V such that $T_n = UV$.*

It is easy to verify that the entries of the above-proposed mosaic preconditioner are generated, in effect, by an asymptotically smooth function (such is any function of the form $||x-y||^{-s}$ with $s > 0$). Thus, the premises of this theorem are fulfilled.

Note that the mosaic-skeleton approximations are constructed only from the matrix entries and the corresponding meshes. The state-of-the-art software manifests a good performance. For example, for the function $f(x,y) = ||x-y||^{-3}$ on $[0,1] \times [0,1]$ the ratio mem $/n$ behaves as $\log^2 n$. Here are some other characteristics (the comression factor is memory used over n^2).

Matrix Size	Compression Factor (%)	Memory (Gb)	Compression Time (sec)
1024	16.22	0.001	1.9
4096	6.39	0.008	11.5
16384	2.39	0.050	62.3
65384	0.83	0.278	307.1
262144	0.27	1.448	1411.5

6 Numerical experiments

In this section, we consider only specific case of domain decomposition, when the internal crossedges are absent. We focus on the arithmetical scalability of the mosaic preconditioner, and the robustness of the trace normalization with respect to the operator coefficients and the meshes. The arithmetical scalability will be presented in terms of CPU time per mosaic preconditioner-vector multiplication, and the robustness will be shown by the condition number of the operator NS, $N = blockdiag\{N_{\hat{M}(k),k}\}$.

6.1 Arithmetical scalability of the mosaic preconditioner

Let $\rho = \varepsilon = 1$ and we consider the simplest Helmholtz operator $-\Delta + I$ with Neumann boundary conditions in a domain Ω which is a union of two similar tetrahedra Ω_i sharing

one common face:

$$\Omega = \left\{ x \mid \sum_{i=1}^{3} |x_i| < \frac{1}{2},\ x_1 > 0, x_2 > 0 \right\},\ m = 2,$$

$$\Omega_1 = \{x \in \Omega, x_3 > 0\},\ \Omega_2 = \{x \in \Omega, x_3 < 0\}.$$

Let the mesh Ω_2^h be a reflection of a tetrahedral mesh Ω_1^h. Therefore, the mesh Ω^h is symmetric. Since we evaluate the dual Schur complement $S = B_1 A_1^{-1} B_1^T + B_2 A_2^{-1} B_2^T$ via factorization of subdomain matrices A_i, $i = 1, 2$, we have to minimize the number of elements in Ω_i^h. On the other hand, in order to illustrate the arithmetical scalability of the preconditioner, we have to increase the order of matrix S, and the number of elements in the mesh $\Theta^h = \bar{\Omega}_1^h \cap \bar{\Omega}_2^h$. A natural solution is a shape regular mesh Ω^h refined geometrically towards Θ such that its trace on Θ, Θ^h, is a uniform mesh. We construct such a mesh by the bisection algorithm [15, 2]. An example of the mesh is shown in Fig.1.

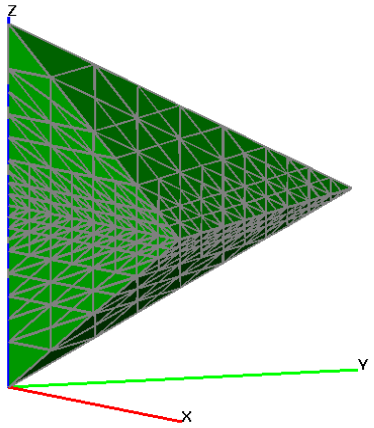

Figure 1: A mesh refined to the common face.

In Table 1 we present the CPU time per multiplication by a mosaic preconditioner as a function of the order of S and number of levels of the mosaic preconditioner. The first four columns point to an almost arithmetical scalability of the mosaic preconditioner. Indeed, the increase of number of nodes in Θ^h by factor 4 results in the increase of CPU

order of S, n_k	153	561	2145	8385	2145	8385
h^{-1}	32	64	128	256	128	256
time, sec	0.016	0.040	0.19	1.04	0.27	2.4
number of levels ?	3	3	3	4	4	3
cond LS	19.3	19.8	19.8	19.9	19.8	19.9

Table 1: CPU time of one evaluation of the mosaic preconditioner L and the condition number of LS.

time by factor 5. The last two columns exhibit two important features of the mosaic preconditioner: 1) the larger order of matrix is, the larger number of levels should be used, 2) on moderate sizes of the matrices, lesser number of levels results in faster evaluation.

6.2 Trace normalization as a block diagonal preconditioner

Now we proceed to testing the robustness of the block diagonal preconditioner $N = blockdiag\{N_{\hat{M}(k),k}\}$. We consider the operator $-\nabla \cdot \rho\nabla + \varepsilon$ with Neumann boundary conditions. The domain Ω is a union of three similar tetrahedra Ω_i sharing one common edge:

$$\Omega = \left\{ x \mid \sum_{i=1}^{3} |x_i| < \frac{1}{2},\ x_1 > 0, \right\} \setminus \{x \mid x_2 > 0, x_3 > 0\},\ m = 3,$$

$$\Omega_1 = \{x \in \Omega, x_2 < 0, x_3 < 0\},\ \Omega_2 = \{x \in \Omega, x_2 < 0, x_3 > 0\},$$

$$\Omega_3 = \{x \in \Omega, x_2 > 0, x_3 < 0\}.$$

A shape regular mesh Ω_1^h is reflected across the common faces between Ω_1 and Ω_2,Ω_3, resp., to obtain the meshes Ω_2^h, Ω_3^h (Fig.2).

In Table 2 we show the condition number of operator NS and the number of PCG iterations needed to reduce the residual by a factor of 10^6, for uniform meshes and different distributions of coefficients ρ_i, ε_i. In Table 3 we present the condition number of operator NS and the number of PCG iterations needed to reduce the residual by a factor of 10^6, for the same coefficients but meshes geometrically refined to a common edge.

As it is seen from Tables 2, 3, both condNS and the number of PCG iterations depend on the mesh very slightly, even for very small meshsteps. The dependence on the coefficient jumps is not observed either, except the case when the global problem is almost decoupled ($\rho_2 = \varepsilon_2 = \rho_3 = \varepsilon_3 = 10^3$, $\rho_1 = \varepsilon_1 = 1$). In the latter case the only off-diagonal blocks of S tie the problem as a whole. Skipping them changes the problem dramatically and the price for that may be seen both in number of iterations and the condition number of the preconditioned dual Schur complement.

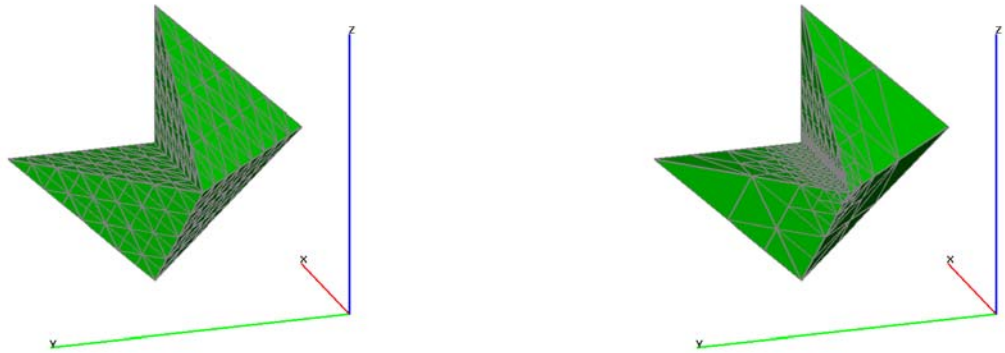

Figure 2: The domain consisting of 3 subdomains and its meshes, uniform (left) and refined (right).

order of S_k, n_k	45	153	561
h^{-1}	16	32	64
$\rho_{1,2,3} = 1$ $\varepsilon_{1,2,3} = 1$	22.7 (21)	23.5 (26)	28.3 (29)
$\rho_{1,2,3} = 1$ $\varepsilon_{1,2,3} = 10^{-4}$	21.7 (15)	22.7 (17)	22.4 (18)
$\rho_1 = \varepsilon_1 = 1$ $\rho_2 = \varepsilon_2 = 10^{-3}$ $\rho_3 = \varepsilon_3 = 10^{3}$	17.3 (20)	17.1 (21)	17.1 (22)
$\rho_1 = \varepsilon_1 = 1$ $\rho_2 = \varepsilon_2 = 10^{-3}$ $\rho_3 = \varepsilon_3 = 10^{-3}$	17.2 (21)	17.3 (22)	17.5 (25)
$\rho_1 = \varepsilon_1 = 1$ $\rho_2 = \varepsilon_2 = 10^{3}$ $\rho_3 = \varepsilon_3 = 10^{3}$	14012 (65)	16706 (87)	18108 (94)

Table 2: Uniform meshes: condition number of NS (number of PCG iterations).

Conclusion

order of S_k, n_k	152	320	660
h^{-1}	64	128	256
$\rho_{1,2,3} = 1$ $\varepsilon_{1,2,3} = 1$	20.6 (24)	21.0 (26)	21.6 (26)
$\rho_{1,2,3} = 1$ $\varepsilon_{1,2,3} = 10^{-4}$	22.4 (15)	22.5 (16)	22.5 (17)
$\rho_1 = \varepsilon_1 = 1$ $\rho_2 = \varepsilon_2 = 10^{-3}$ $\rho_3 = \varepsilon_3 = 10^{3}$	12.2 (20)	12.1 (22)	12.2 (22)
$\rho_1 = \varepsilon_1 = 1$ $\rho_2 = \varepsilon_2 = 10^{-3}$ $\rho_3 = \varepsilon_3 = 10^{-3}$	16.1 (24)	15.4 (24)	16.2 (23)
$\rho_1 = \varepsilon_1 = 1$ $\rho_2 = \varepsilon_2 = 10^{3}$ $\rho_3 = \varepsilon_3 = 10^{3}$	13774 (87)	13626 (108)	13468 (114)

Table 3: Locally refined meshes: condition number of NS (number of PCG iterations).

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