Piecewise Separable Matrices

Eugene Tyrtyshnikov [1]

Institute of Numerical Mathematics, Russian Academy of Sciences,
Gubkina Street, 8, Moscow 119991 (e-mail: tee@inm.ras.ru)

Abstract. Piecewise separable matrices are introduced as a natural generalization of semiseparable and profile-low-rank matrices. It is shown that all matrices from this class possess linear matrix-by-vector complexity.

1 Introduction

Separability is a pervasive concept in functions of two variables, where it means (as a rule, approximate) separation of the variables, and in matrices, where it means approximation by an outer product of two vectors (also known by the name of dyad, skeleton, or matrix of rank 1). Usually, separabilty is related to a certain "smoothness" of data and might not extend to the whole of data, e.g. all entries of an invertible matrix. However, in such cases there could be blocks of the entries where this property still applies (cf [7, 12, 13, 14]).

Also, it was discovered long ago that inverses to banded matrices possess separability in domains of a more complicated shape than rectangular blocks. The first reference to the case of tridiagonal matrices with nonzero boundary diagonals is probably Gantmacher and Krein [8] (see also [1, 2] and a thorough look back into the history in [17]).

In the latter case, the inverse has its lower and upper triangular parts as those cut of from some rank-1 matrices, generally different but coinciding along the main diagonal. This is a prototype of a semiseparable matrix. An important generalization arises by adding an arbitrary diagonal (see [3, 5,

[1]Supported in part by the Russian Fund of Fundamental Research (Grants N 04-07-90336, N 05-01-00721).

6, 9]). Another generalizations appears from the claim that any submatrix located totally in one of the strictly lower or upper triangular parts is of rank 1 (so-called quasiseparable [4] or weakly semiseparable [15] matrices).

One more generalization used in the study of Kronecker-product approximation is that of profile-low-rank matrices [12] (instead of formal definition, have a look at Fig. 1).

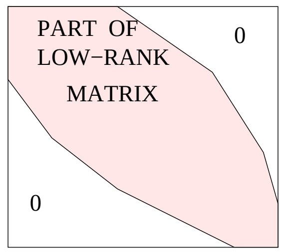

Figure 1: A profile-low-rank matrix.

The purpose of this note is some remarks on the matrix-by-vector complexity for matrices with the above and a little generalized separability properties. This topic is directly addressed in the studies of multilevel matrices with separable blocks resulting there in $O(n \log^{\alpha} n)$ estimates, $\alpha > 0$ depending on the context (cf [12, 14, 16]). It is easy to see that a straightforward recursive divide-and-conquer approach to a semiseparable matrix of order n delivers a $O(n \log n)$ procedure. However, this certainly ignores some of this structure, because an $O(n)$ algorithm is almost equally immediate.

Indeed, consider a matrix A with the entries

$$a_{ij} = \begin{cases} u_i v_j, & i \geq j, \\ 0\,, & i < j. \end{cases}$$

Then $y = Ax$ with $y = [y_i]$, $x = [x_i]$ can be obtained in the following way:

(1) Initialize $w_0 = 0$.

(2) For $i = 1, 2 \ldots, n$ compute

$$w_i = w_{i-1} + v_i x_i.$$

(2) For $i = 1, 2 \ldots, n$ compute

$$y_i = u_i w_i.$$

In [12] it was observed that the linear complexity holds also for profile-low-rank matrices. Here we introduce a wider class of matrices with arbitrary curvilinear "profiles" and still with the same linear complexity. This kind of structure appears naturally in matrices composed of the values of a piecewise smooth function on some grids (cf.[12]).

2 Piecewise separability

Definition. A matrix $A = [a_{ij}]$ of order n is called *piecewise separable* if

$$a_{ij} = \begin{cases} u_i^0 v_j^0, & \tau_i^0 < j \le \tau_i^1, \\ u_i^1 v_j^1, & \tau_i^1 < j \le \tau_i^2, \\ \quad \dots & \quad \dots \\ u_i^p v_j^p, & \tau_i^p < j \le \tau_i^{p+1}, \end{cases}$$

where u_i^k, v_i^k are given values and τ_i^k are given indices subject to the following inequalities:

$$0 = \tau_i^0 \le \tau_i^1 \le \ \dots \ \le \tau_i^p \le \tau_i^{p+1} = n, \qquad i = 1, \dots, n.$$

Clearly, the storage for such a matrix is $O(pn)$.

When $p = 1$ and $a_{ij} = 0$ for $\tau_i^1 < j \le n$, we call A a *standard one-piece* matrix. Obviously, any piecewise separable matrix with p boundaries can be written as a sum of $O(p)$ standard one-piece matrices. Hence, from the point of matrix-by-vector multiplication, it is sufficient to investigate only the case of standard one-piece matrices.

Thus, let A be a standard one-piece matrix. Then, denote by

$$0 = \tau_0 \ \le \ \tau_1 \ \le \ \tau_2 \ \le \ \dots \ \le \ \tau_n$$

the same indices $\tau_1^1, \tau_2^1, \dots, \tau_n^1$ taken in the non-decreasing order. We can write $\tau_i = \tau_{\sigma(i)}^1$ for an appropriate permutation σ. Consider now a new standard one-piece matrix $B = PA$, where $P = [p_{ij}]$ is a permutation matrix defined by σ as follows:

$$P = \begin{cases} 1, & j = \sigma(i), \\ 0, & \text{otherwise.} \end{cases}$$

Also, set $u_i = u^1_{\sigma(i)}$ and $v_i = v^1_i$. Then B is a standard one-piece matrix with generating values u_i, v_i and indices τ_i. We can reduce multiplication by A to multiplication by B and then by $P^{-1} = P^\top$.

Algorithm. *Computation of $y = Ax$ is done by the following steps:*

(1) *Initialize $w_0 = 0$.*

(2) *For $i = 1, 2 \ldots, n$ compute*

$$w_i \;=\; w_{i-1} + \sum_{j=\tau_{i-1}+1}^{\tau_i} v_j x_j.$$

(3) *For $i = 1, 2 \ldots, n$ compute*

$$y_{\sigma(i)} \;=\; u_i w_i.$$

The operation count finishes with an $O(n)$ estimate because of the evident inequality

$$\sum_{i=1}^{n} (\tau_i - \tau_{i-1}) \;\leq\; n.$$

This proves (together with the previous remarks) that any piecewise separable matrix with p boundaries can be multiplied by a vector in $O(pn)$ operations.

3 Piecewise quasiseparable matrices

It is interesting to note that recently introduced quasiseparable [4] and weakly semiseparable [15] matrices can be also considered as piecewise separable matrices.

Quite formally, we can introduce a notion of *piecewise quasiseparable* matrices: consider the index domains

$$\mathcal{I}^k = \{(i,j) : \tau_i^{k-1} < j \leq \tau_i^k, \quad 1 \leq i \leq n\}$$

and require that any submatrix with indices $(i,j) \in \mathcal{I}^k$ is of a bounded rank (e.g. 1). It is likely, all the same, that there should be some additional requirements on the boundaries for this class to be really useful.

Consider first the case of one boundary defined by $\tau_i = \tau_i^k$, and assume that A is such that all submatrices in each of the two index domains $\mathcal{I}_1$ and $\mathcal{I}_2$ have rank less than or equal to 1. By row permutations, we can transform A into a matrix with a monotone boundary. Thus, without loss of generality, assume that $\tau_1 \le \tau_2 \le \ldots \le \tau_n$. In the upper domain $\mathcal{I}_2$, take up any 2×2 block

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

and suppose that $\alpha \neq 0$. Then, $\beta = 0$ or $\gamma = 0$ if and only if $\delta = 0$. This simple observation results in the following. Let

$$a_{1j_1} \neq 0 \quad \text{and} \quad a_{1j} = 0, \; j > j_1.$$

Then $a_{ij} = 0$ for all $(i, j) \in \mathcal{I}_2$, $j > j_1$. Thus, we can introduce an additional monotone boundary in the upper part of A so that the third domain consists of zeroes and the middle domain consists of the two subdomains belonging to different diagonal blocks in a 2×2 partitioning of A. The first of these subdomains is clearly a part of a matrix of rank 1. The second one may undergo a similar partitioning. Applying the same procedure recursively, we come up with one additional boundary in the upper domain. The part of A between the original and new boundaries can be embedded into a matrix of rank 1, and the part upper the new boundary contains only zeroes. Similarly, an additional boundary can be constructed in the lower domain $\mathcal{I}_1$. All in all, the two additional boundaries convert A into a piecewise separable matrix with three boundaries.

The same construction can be applied to a more general case of a quasiseparable matrix with any number of monotone boundaries. Thus, a quasiseparable matrix with p monotone boundaries can be described by $O(pn)$ generating parameters and multiplied by a vector in $O(pn)$ operations.

An open question is whether the number of generating parameters and matrix-by-vector complexity remain linear for an arbitrary piecewise separable matrix. Also, notice a problem of finding generating values and indices from a given sample of the matrix entries. Theory and algorithms for some particular cases are developed in [7, 10, 11, 16], but better insight and extensions are still in need and in line with very interesting applications.

References

[1] E. Apslund. Inverses of matrices $\{a_{ij}\}$ which satisfy $a_{ij} = 0$ for $j > i+p$, *Math. Scand.*, 7 (1): 57–60 (1959).

[2] W. Barrett and P. Feinsilver. Inverses of banded matrices, *Linear Algebra Appl.*, 41: 111–130 (1981).

[3] Y. Eidelman and I. Gohberg, Fast inversion algorithms for diagonal plus semiseparable matrices, *Integral Equations and Operator Theory*, 27 (2): 165–183 (1997).

[4] Y. Eidelman and I. Gohberg, On a new class of structured matrices, *Integral Equations and Operator Theory*, 34: 293–324 (1999).

[5] D. Fasino and L. Gemignani, Structural and computational properties of possibly singular semiseparable matrices, *Linear Algebra Appl.*, 340: 183–198 (2001).

[6] D. Fasino, N. Mastronardi, and M. Van Barel, Fast and stable algorithms for reducing diagonal plus semiseparable matrices to tridiagonal and bidiagonal form, *Contemporary Mathematics*, 323: 105–118 (2003).

[7] J. M. Ford, E. E. Tyrtyshnikov, Combining Kronecker product approximation with discrete wavelet transforms to solve dense, function-related systems, *SIAM J. Sci. Comput.*, Vol. 25, No. 3, pp. 961–981 (2003).

[8] F. R. Gantmacher and M. G. Krein, Sur les matrices oscillatoires et completement non negatives, *Compositio Mathematica*, 4: 445–476 (1937).

[9] I. Gohberg, T. Kailath, and I. Koltracht, Linear complexity algorithms for semiseparable matrices, *Integral Equations and Operator Theory*, 8 (6): 780–804 (1985).

[10] S. A. Goreinov, E. E. Tyrtyshnikov, N. L. Zamarashkin, A Theory of Pseudo-Skeleton Approximations, *Linear Algebra Appl.*, 261: 1–21 (1997).

[11] S. A. Goreinov, E. E. Tyrtyshnikov, The maximal-volume concept in approximation by low-rank matrices, *Contemporary Mathematics*, Vol. 208, 47–51 (2001).

[12] W. Hackbusch, B. N. Khoromskij, and E. E. Tyrtyshnikov, *Hierarchical Kronecker tensor-product approximations*, Max-Panck-Institut fur Mathematatik in den Naturwissenschaften, Leipzig, Preprint No. 35 (2003).

[13] E. E. Tyrtyshnikov, Mosaic ranks and skeletons, *Lecture Notes in Computer Science 1196: Numerical Analysis and Its Applications. Proceedings of WNAA-96.* Springer-Verlag, pp. 505-516 (1996).

[14] E. E. Tyrtyshnikov, Mosaic-skeleton approximations, *Calcolo*, 33 (1–2), 47–57 (1996).

[15] E. E. Tyrtyshnikov, Mosaic ranks for weakly semiseparable matrices, *Notes on Numerical Fluid Mechanics*, Vol 73, pp. 36–41 (2000).

[16] E. E. Tyrtyshnikov, Incomplete cross approximation in the mosaic-skeleton method, *Computing*, 64, no. 4, 367–380 (2000).

[17] R. Vandebril, G. Golub, M. Van Barel and N. Mastronardi, *A bibliography on semiseparable matrices*, `http://www.cs.kuleuven.ac.be/ mase/` (2004).