On the approximation of Toeplitz matrices by a sum of circulant and low-rank matrix *

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1. Introduction. A matrix $T = [a_{ij}]_{ij=1}^{n}$ is said to be Toeplitz, if $a_{ij} = t_{i-j}$. To solve linear systems with Toeplitz matrices it is convenient to use some iterational method. However, to achieve fast convergence, preconditioning should be used. It was first proposed in [1] to use circulant preconditioners, which were studied later in [2, 3] and in a large number of other papers. All proofs for fast(superlinear) convergence of iterational methods are based on the decomposition [3]

$$T = C + R + E,$$

where C is a circulant matrix, $||E|| \leq \varepsilon$, R — a matrix of rank $r = r(\varepsilon, n) \ll n$. Rank estimates obtained in [3] and some other papers are of form $r = \mathcal{O}(1)$ or $r = o(n)$; but unfortunately, $r \sim \frac{1}{\varepsilon^{\alpha}}$, $\alpha > 0$. In this paper it is proved that in typical cases (including all examples in papers on the construction of superlinear preconditioners) a circulant matrix can be chosen so that $r(\varepsilon, n) = \mathcal{O}((\log \varepsilon^{-1} + \log n) \log \varepsilon^{-1})$.

2. Toeplitz matrices with rational symbols. A matrix T is said to be associated with a symbol f, if

$$t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-itk} dt.$$

Lemma 1. *Let T be a lower-triangular Toeplitz matrix with the first column $t_k = \alpha \rho^k$. Then $T = C + R$, where C is a circulant matrix and R is a matrix of rank 1*

Proof. It is sufficient to take R as a rank-1 Toeplitz matrix

$$R = [r_{i-j}], \quad r_k = \frac{\alpha \rho^k}{1 - \frac{1}{\rho^n}}, \quad k = -n+1, ..., n-1.$$

It is straightforward to check that the matrix $C = T - R$ is circulant.

*) This work was supported by a Russian Fund of Basic Research, (grant 05-01-00721).

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Corollary 1. *If a Toeplitz matrix T has a symbol*

$$f(z) = \frac{1}{1-\rho z}, \quad z = e^{it}, \quad |\rho| \neq 1,$$

then $T = C + R$, where C is a circulant matrix, a R is a matrix of rank 1.

Lemma 2. *Let T be a lower-triangular Toeplitz matrix with the first column* $t_k = k^q$, q is a natural number. *Then $T = C + R$, where C is a circulant matrix, and* $\text{rank} R \le q + 2$.

Proof. Let us choose R as a Toeplitz matrix

$$R = [r_{i-j}], \quad r_k = p(k),$$

where p is a polynomial of degree $q + 1$ satisfying

$$p(k) - p(k - n) = k^q.$$

Obviously, the rank of matrix R doesn't exceed $q+2$, and the matrix $C \equiv T - R$ is circulant because

$$c_k - c_{k-n} = t_k - t_{k-n} - r_k - r_{k-n} = 0.$$

Theorem 1. *Let a Toeplitz matrix T be associated with a rational trigonometric symbol of form*

$$f(z) = P(z) + \frac{Q(z)}{L(z)}, \quad z = e^{it},$$

where P, Q, L are polynomials, L has no roots on a unit circle, the degree of Q is not greater than the degree of L and they have no common roots. Then

$$T = C + R,$$

where C is a circulant matrix, and $\text{rank}(R) \le \deg P + \deg L + 1$.

Proof. Split $\frac{Q(z)}{L(z)}$ in elementary fractions and use Corollary 1.

2. Toeplitz matrices with logarithmic singularities of symbols.

Lemma 3. *Let T be a lower-triangular Toeplitz matrix with the first column*

$$t_k = \begin{cases} 0, & k = 0; \\ \rho^k k^{-\alpha}, & k = 1, \dots, n-1, \;\; \alpha > 0. \end{cases}$$

Then for each ε there exists a circulant matrix C and a matrix R of rank r such that

$$|(T - C - R)_{ij}| \le |T_{ij}|\varepsilon,$$

and

$$r \leq \log \varepsilon^{-1}[c_0 + c_1 \log \varepsilon^{-1} + c_2 \log n],$$

where c_0, c_1, c_2 *depend only on* α.

Proof. For each ε there exist f_m, w_m such that [4]

$$|k^{-\alpha} - \sum_{m=1}^{r} w_m e^{-f_m k}| \leq k^{-\alpha}\varepsilon, \qquad r \leq \log \varepsilon^{-1}[c_0 + c_1 \log \varepsilon^{-1} + c_2 \log n].$$

It is now left to apply Lemma 1.

Corollary 3. *Let a Toeplitz matrix* T *be associated with the symbol*

$$f(z) = \log(z - \zeta), \quad z = e^{it}, \quad |\zeta| = 1.$$

Then for each ε *there exists a circulant matrix* C *and a matrix* R *of rank* r *such that*

$$|(T - C - R)_{ij}| \leq |T_{ij}|\varepsilon,$$

and

$$r \leq \log \varepsilon^{-1}[c_0 + c_1 \log \varepsilon^{-1} + c_2 \log n].$$

Corollary 4. *Let a Toeplitz matrix* T *be associated with the symbol*

$$f = (z - \zeta)^{\alpha} \log(z - \zeta), \quad z = e^{ix}, \quad |\zeta| = 1, \quad \alpha \in \mathbb{N}.$$

Then for each ε *there exists a circulant matrix* C *and a matrix* R *of rank* r *such that*

$$|(T - C - R)_{ij}| \leq |T_{ij}|\varepsilon,$$

and

$$r \leq \log \varepsilon^{-1}[c_0 + c_1 \log \varepsilon^{-1} + c_2 \log n] + 2\alpha.$$

The results of this part are summarized in the following

Theorem 2. *Let a Toeplitz matrix* T *be associated with a piecewise-analytic symbol of form*

$$f = g + \sum_{\alpha=0}^{l} \sum_{k=0}^{m} A_{k\alpha}(z - \zeta_k)^{\alpha} \log(z - \zeta_k), \quad z = e^{it}, |\zeta_k| = 1,$$

where g *is analytic in a disk containing* $|z| = 1$. *Then for each* ε *there exists* C *and a matrix* R *such that*

$$|(T - C - R)_{ij}| \leq |T_{ij}|\varepsilon,$$

and

$$\operatorname{rank}(R) \leq \log \varepsilon^{-1}[c_0 + c_1 \log \varepsilon^{-1} + c_2 \log n] + c_3,$$

and c_0, c_1, c_2, c_3 *do not depend on* n *and* ε.

References

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