Semidefinite programming. Part 1

Notations

- S^n – set of real symmetric $n \times n$ matrices;
- $S^n_{\succeq 0}$ – set of real symmetric nonnegative matrices;
- $A \bullet B = trace(A^T B)$ – inner scalar product for matrices A, B of the same size;
- Primal SDP

$$\begin{aligned} p^* = \ & \text{maximize} && C \bullet X \\ & \text{subject to} && A_i \bullet X = b_i,\ i = 1 \dots m \\ & && X \succeq 0 \end{aligned} \tag{1}$$

- Dual SDP

$$\begin{aligned} d^* = \ & \text{minimize} && b^T y \\ & \text{subject to } y \in \mathbb{R}^m : && \textstyle\sum_{i=1}^m y_i A_i - C \succeq 0 \end{aligned} \tag{2}$$

  with $b = [b_1 \dots b_m]^T \in \mathbb{R}^m$.

Exercises 1-4: nonnegative matrices;
Exercises 5-10: nonnegative matrices and $\bullet$ product (needed for theory);
Exercises 11-16: SDP, dual SDP.

1. $A \in \mathbb{C}^{n\times n}$. $\forall x \in \mathbb{C}^n :\ (Ax, x) \geq 0$. Prove that $A = A^*$.
   $A \in \mathbb{R}^{n\times n}$. $\forall x \in \mathbb{R}^n :\ (Ax, x) \geq 0$. Disprove that $A = A^T$.
2. $A \in S^n_{\succeq 0}$, $rankA = 1$. Prove that $\sqrt{A} = \alpha A$, and find α.
3. $A \in S^n_{\succeq 0}$ and $x, y \in \mathbb{R}$. Prove that $(Ax, x)(A^{-1}y, y) \geq (x, y)$.
4. $A, B \in S^n_{\succeq 0}$. Prove that $A \otimes B$, $A \odot B \succeq 0$ (Kroneker and Hadamar products).
5. $A \in S^n$. Prove that $A \succeq 0$ if and only if for every $B \in S^n_{\succeq 0} : A \bullet B \geq 0$.
6. $A, B \in S^n_{\succeq 0}$, $rankA = r$, $A = VV^T$ for some $V \in \mathbb{R}^{n\times r}$, and $ker(A) \subseteq ker(B)$. Prove that $B = VQV^T$ for some $Q \in \mathbb{R}^{r\times r}$.
7. $A \in S^n_{\succeq 0}$, $V \in \mathbb{R}^{n\times k}$, and $A \bullet (VV^T) = 0$. Prove that $AV = 0$.

8. $A, B \in S^n_{\succeq 0}$, and $A \bullet B = 0$. Prove that $rankA + rankB \leq n$ and $AB = 0$.

9. $A, B \in S^n_{\succeq 0}$. Prove that $Im(A + B) = Im(A) + Im(B)$.

10. $A \in S^n_{\succeq 0}$, and $|A_{ij}| \leq 1$ for all i, j. Prove that $arcsin(A) \succeq A$ (elementwise $arcsin$).

11. (Theorem) Prove that for every feasible solution X of (1) and for every feasible solution y of (2): $C \bullet X \leq b^T y$. Thus $p^* \leq d^*$, and if for some feasible X and $y:$ $C \bullet X = b^T y$, then $p^* = d^*$.
(Hint: consider $X \bullet (\sum_i y_i A_i - C)$)

12. Prove that

$$\begin{aligned} \lambda_{max}(C) = \max \quad & C \bullet X \\ \text{subject to} \quad & I \bullet X = 1, \\ & X \succeq 0. \end{aligned}$$

(Without duality or $rank = 1$ condition).

13. Let X be an optimal solution of SDP-MaxCut problem, and $rank\, X = 1$. Prove that the resulting set S does not depend on the particular choice of matrix R and random vector p.

14. Write primal and dual SDP for $\lambda_{min}(C)$ of some $C \in S^n$.

15. Consider SDP:

$$\begin{aligned} \text{maximize} \quad & C \bullet X \\ \text{subject to} \quad & I \bullet X = k, \\ & I \succeq X \succeq 0. \end{aligned} \tag{3}$$

for some $C \in S^n$. Prove that maximum equals $\lambda_1 + \ldots + \lambda_k$ (for k largest eigenvalues λ_i of C).

16. Write the dual SDP for problem (3).