STUDENT PROBLEMS ON MATRICES, TENSORS, COMPUTATIONS

1. Two systems $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ of linearly independent vector-columns determine a matrix $A = u_1 v_1^\top + \ldots + u_r v_r^\top$. Prove that $\operatorname{rank} A = r$.

2. Any k vector-columns in each of the two systems $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ are linearly independent. Prove that any $2k-1$ matrices among $u_1 v_1^\top, \ldots, u_r v_r^\top$ are linearly independent.

3. Let A be a real symmetric matrix of order n, and the maximal value of the function $f(u,v) = |u^\top A v|$ of vectors $u, v \in \mathbb{R}^n$ subject to the condition $||u||_2 = ||v||_2 = 1$ is attained on some vectors $u = x,\ v = y$. Prove that
$$|x^\top A y| = |x^\top A x| = |y^\top A y|.$$

4. Let a be an arbitrary 3-tensor over a field $\mathbb{P}$, and let R_1, R_2, R_3 be its Tucker ranks in the order $R_1 \le R_2 \le R_3$. Then
$$R_3 \le \operatorname{trank}_{\mathbb{P}}(a) \le R_1 R_2.$$

5. For the maximal rank of tensors over an arbitrary field, prove that $\operatorname{mrank}(m,n,q) = mn$ if $q \ge mn$.

6. Let a tensor of size $n \times n \times 3$ be defined by three $n \times n$ matrices of its sections A_1, A_2, A_3 in the third dimension. Prove that if any two of these matrices are diagonal, then the canonical tensor rank of this tensor does not exceed $2n$.

   Atkinson and Lloyd claimed that this result holds true for complex tensors of size $n \times n \times 4$ as well; in the case of q sections the rank is upper bounded by $\lceil q/2 \rceil\, n$. Try to ponder on the proof.

7. Prove that the set of all complex $m \times n$ matrices of rank not greater than $k \le \min(m,n)$ is an irreducible algebraic variety of dimension $k(m+n-k)$.

8. Let L be an arbitrary linear subspace in the space of complex matrices of size $m \times n$, and let $1 \le k \le \min(m,n)$. Prove that if $\dim L \ \ge\ d \equiv (m-k)(n-k)+1$, then L contains a nonzero matrix of rank not greater than k.

   Such a matrix may not exist in L in the case $\dim L < d$. Let $m = n = 3$ and $k = 1$, then $d = 5$. Give an example of 4 matrices of order 3 for which their linear span does not contain a nonzero matrix of rank 1.

9. Prove that the set of all complex tensors of size $m \times n \times q$ with canonical rank not greater than 2 is not closed for any $m, n, q \ge 2$.

10. For the generic rank, prove that $\operatorname{grank}(3,3,3) = 5$.