Dear students the rules are simple. You can pass problems oraly or send Sergey within email.

First, pass through at least 3 easy problems from seminars by Sergey Matveev. Then you can follow the list by professor Tyrtyshnikov:

- 3 problems from seminars + 3 problems from lectures – "OK"
- 3 problems from seminars + 4 problems from lectures – "GOOD"
- 3 problems from seminars + 5 problems from lectures – "EXCELLENT"

Thus, at least 6 problems have to be solved – 3 from seminars and 3 from lectures.

Problems for the lecture course of Professor Tyrtyshnikov

1. Find the singular value decomposition for the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

2. Prove that $||A||_F \leq \sqrt{\text{rank}\, A}\, ||A||_2$.

3. Let $\sigma_1 \geq \ldots \geq \sigma_n$ be the singular values of a matrix $A \in \mathbb{R}^{n\times n}$. Prove that
$$\min_{\text{rank}\, B \leq k} ||A - B||_F = \sqrt{\sum_{\alpha \geq k+1} \sigma_\alpha^2}.$$

4. Prove that any tensor of size $n \times n \times (n^2 + 2018)$ can be represented by a sum of n^2 pure tensors, and there is a tensor that cannot be represented by a sum with smaller number of pure tensors.

5. A tensor of size $3 \times 3 \times 2$ is defined by its two sections
$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Prove that it cannot be represented by a sum with less than 4 pure tensors, and can be written as a sum of exactly 4 pure tensors.

6. A tensor with d indices has the elements of the form
$$a_{i_1 \ldots i_d} = \sin(i_1 + \ldots + i_d).$$
Prove that this tensor has a tensor-train decompostition with all tensor-train ranks not exceeding 2.

7. Any k vector-columns in each of the two systems $u_1, \ldots, u_r$ and $v_1, \ldots, v_r$ are linearly independent. Prove that any $2k - 1$ matrices among $u_1 v_1^\top, \ldots, u_r v_r^\top$ are linearly independent.

Problems for the seminars "Applications of tensor trains to numerical simulation" by Sergey Matveev

1. Let R be row basis for matrix A and C be column basis. B stands on intersection of row and column bases. Prove that $A = C\ B^{-1}\ R$.

2. Implement matrix cross-sampling algorithm and compare its CPU-times with SVD for square matrices of size $N = 2^{10}, 2^{12}, 2^{14}$ for matrices generated by the following equation
$$A_{i,j} = i^a j^b + i^b j^a$$
for $a = 0.1$, $b = 0.2$. Convergence parameter $\varepsilon = 10^{-5}$. How many steps were made by cross algorithm?

3. Let $C(i, j, k)$ be a sum of R pure tensors. Prove that
$$f_k = \sum_{i=1}^{N} \sum_{j=1}^{N} C(i, j, k) n_i n_j$$
can be evaluated in $O(NR)$ operations. What would be the complexity of this operations for $C(i, j, k)$ represented as tensor train with maximal rank R?

4. Prove that cost of evaluation of d-dimensional array represented as tensor train with maximal rank R is $O(dR^2)$ operations.