Nonnegative and spectral matrix theory with applications to network analysis — Final test

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This document contains the problems for the final evaluation of the course "Nonnegative and spectral matrix theory with applications to network analysis", Rome-Moscow School on Matrix Methods and Applied Linear Algebra 2018. You are requested to solve as many as possible of them. Problems marked with a star ($\star$) are a bit more difficult than the others. Please send your solutions before September 3, 2018 to the email address `dario.fasino@uniud.it`. Moreover, for a fair evaluation, specify your current position (graduate student, PhD student, PostDoc...).

1. **The starred star graph.** For any given integers $p, q \geq 1$, let $\mathcal{G}$ be the undirected graph having $1+p+pq$ nodes and $p+pq$ undirected edges defined as follows:

   - There is a *root* node, which is connected to p *star* nodes;
   - every *star* node is connected to the root node and to q *leaf* nodes.

   Compute the Bonacich centrality index for the nodes of this graph. Use the preceding result to prove that the Bonacich index of a node is not an increasing function of its degree. Under what condition on p, q the root node doesn't get the highest score?

   Hint: Use graph automorphisms to reduce the solution of $Ax = \rho(A)x$ to three scalar equations.

2. **A characterization of Perron vectors.** Let $A \geq O$ be irreducible. If $x \geq 0$ is any eigenvector of A and $Ax = \lambda x$ then $\lambda = \rho(A)$.

3. **A sufficient condition for primitivity.** Let $A \geq O$. Prove that if $\mathcal{G}_A$ is strongly connected and has at least one loop then A is primitive. Find an upper bound to the least integer k such that $A^k > O$.

4. $\star$ **Sharpening Dietzenbacher's theorem.** Prove the following corollary to Theorem 2.9: *In the same hypotheses and notations of Theorem 2.9, if $\rho = \hat{\rho}$ then either x and $\hat{x}$ are multiple of each other or*

   $$\forall i \in \mathcal{I}, \qquad \min_{j=1\ldots n} \frac{\hat{x}_j}{x_j} < \frac{\hat{x}_i}{x_i} < \max_{j=1\ldots n} \frac{\hat{x}_j}{x_j}.$$

   Hint: Recall that a matrix is irreducible when the associated graph is strongly connected.

5. $\star$ **A condition for non-nilpotency.** Let $A \geq O$. Prove that $\rho(A) > 0$ if and only if there is at least one closed walk in $\mathcal{G}_A$.

   Note: a closed walk is a walk which starts and terminates at the same node.

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