Let's start with matrices. Matrix is a 2-dimensional array $A = (a_{ij}), i = 1..n, j = 1..m$. Let's define matrix rank as lest $r = rk(A)$ such that there exists vectors $a_1, ..., a_r \in \mathbb{C}^n, b_1, ..., b_r \in \mathbb{C}^m$:

$$a_{ij} = \sum_k^r a_{ki} b_{kj}$$

or in more compact way:

$$A = \sum_k^r a_i b_i^T$$

I will denote this sum in a more abstract way as:

$$A = \sum_k^r a_i \otimes b_i$$

Such decomposition called as skeleton decomposition. The meaning of "$\otimes$" would be explained further. "$a \otimes b$" - called as tensor product of vector a and b. And a matrix has rank one iff $A = a \otimes b$ for some vector a and b.

Definition 1. *$c = a \otimes b$ called as Kronecker (or tensor) product of vector a and b. In coordinates: $c_{ij} = a_i b_j$.*

You can easily prove that Kronecker product associative and non commutative. One of the main facts about matrices and rank is existence of best approximation via matrices of smaller rank.

Theorem 1. *Let $A \in \mathbb{C}^{n \times m}$ then for all r there exists matrix B of rank lesser of equal to r s.t. $|A - B| = \min_{rk(B) \leq r} |A - B|$, where $|A|$ denotes eucledian norm. And B can be obtained via SVD-decomposition.*

Excercise 1. *Prove that if $rk(A_n) \leq r$ and $lim A_n = A$ then $rk(A) \leq r$ for all large n*

It's time to move from matrices to tensors.

Definition 2. *3-dimensional array $t = (t_{ijk}), i = 1..n, j = 1..m, k = 1..q$ is called tensor. Vector space of all possible tensors denoted as $\mathbb{C}^{n \times m \times q}$ with component-wise addition and scalar multiplication.*

Now we can generalize definition of matrix rank to tensor rank.

Definition 3. *Least number r s.t. there exists vectors $a_1, ..., a_r, b_1, ..., b_r, c_1, ..., c_r$:*

$$t_{ijk} = \sum_s^r a_{si} b_{sj} c_{sk}$$

or using "$\otimes$" notation:

$$t = \sum_s^r a_s \otimes b_s \otimes c_s (*)$$

is called tensor rank and denoted as $r = rk(T)$.
(*) - such sum called as canonical decomposition of the tensor t.
$d = a \otimes b \otimes c$ - Kronecker product of three vectors, in coordinates: $d_{ijk} = a_i b_j c_k$.

We can see that definitions of matrix and tensor rank are very similar. Essential question is: Is exercise 1 and theorem 1 hold for tensors? Unfortunately the answer is no.

Excercise 2. *Prove that ex.1 fails for tensor* $t = a \otimes a \otimes b + b \otimes a \otimes a + a \otimes b \otimes a$ *where* a, b *linearly independent vectors assuming that* $rk(t) = 3$. *(Hint: take a view at tensor* $(a + \epsilon b) \otimes (a + \epsilon b) \otimes (a + \epsilon b)$ *where* ϵ *is an arbitrary small number)*

Excercise 3. *Prove that* t *from ex.2 has rank equals 3 in case* $a = (1, 0), b = (0, 1)$.

Excercise 4. *Prove ex.3 in general case.*

Excercise 5. *Prove that if* $lim t_n = t \neq 0$ *and* $rk(t_n) = 1$ *then* $rk(t) = 1$

And here arises new definition that should fix this problems.

Definition 4. *Least* r *s.t. there exists sequence* t_n *s.t.* $rk(t_n) = r$ *and* $lim t_n = t$ *is called as border rank of tensor* t *and denoted as* $brk(t)$

Border rank and rank of a tensor are essential generalizations of matrix rank.

Statement 1. *For every tensor* t *holds:* $brk(t) \leq rk(t)$.

Proof. Let $t_n = t, rk(t_n) = rk(t)$ and $lim t_n = t$, hence $brk(t) \leq rk(t)$. □

It can be shown with use of algebraic geometry that border rank coincide with rank mostly everywhere. But unfortunately that's not the concern of our course. We have two notions of tensor rank. But we need some algebraic tools to compute it or at least to bound it. What's the best way? Introduce some more notions of rank that can be easily computed.
First we need to create linear maps from tensor t. Let's write some decomposition of $t \in \mathbb{C}^{n \times m \times q}$:

$$t = \sum_i a_i \otimes b_i \otimes c_i$$

Definition 5. *Let* $x \in \mathbb{C}^n$. *The first flattering of tensor* t *is*

$$t_1(x) := \sum_i (a_i x^T) b_i \otimes c_i, t_1 : \mathbb{C}^n \to \mathbb{C}^{m \times q}$$

In the same way we can define second and third flattering.

Excercise 6. *Prove that first flattering is independent from choose of tensor decomposition i.e. definition is correctly defined. (Hint: prove that* $(t_1(x))_{jk} = \sum_i t_{ijk} x_i$*)*

Definition 6. *Flatterings are linear maps. Hence we can compute their matrix ranks.*

$$rk_i(t) := rk(t_i)$$

This ranks called as flattering ranks.

From previous exercise follows that flattering ranks are well defined. Essential questions is how they relate to border and canonical ranks?

Statement 2. *For every tensor t we have*

$$rk_i(t) \leq brk(t) \leq rk(t)$$

Proof. Let $t_n \to t, rk(t_n) = brk(t)$. Let's write canonical decomposition for every t_n: $t_n = \sum a_i^n \otimes b_i^n \otimes c_i^n$. Then $t_1^n = \sum_i a_i^n \otimes (b_i^n \otimes c_i^n)$ converges to t_1. Then by exercise 1 we have that $rk(t_1) \leq rk(t_1^n) \leq rk(t_n) = brk(t)$. □

Definition 7. $rk_{mult} := (rk_1, rk_2, rk_3)$ *is called multilinear rank.*

Statement 3. $rk(t) = 1$ *iff* $rk_{mult}(t) = (1, 1, 1)$.

Proof. $\Leftarrow$ Write down $t_1 = a \otimes t', t' = \sum_i b_i \otimes c_i$ - skeleton decomposition. Then $t = \sum_i a \otimes b_i \otimes c_i$. From matrices we know that $\{b_i\}$ and $\{c_i\}$ are linearly independent sets. Then $\{a \otimes b_i\}$ is linearly independent set. Then for $t_3 = \sum_i (a \otimes b_i) \otimes c_i$ we have that $rk(t_3) = rk(t')$. But $rk(t_3) = rk_3(t) = 1$. Then $t' = b \otimes c$ and $t = a \otimes b \otimes c$. □

Excercise 7. *Compute multilinear ranks for tensor from exercise 2.*

Why do we need canonical decomposition? Suppose that you have a tensor $t \in \mathbb{C}^{n \times n \times n}$ s. t. $rk(t) << n, n >> 1$. In order to store the whole tensor in the memory of your computer you need n^3 space. But if you store only canonical decomposition you'll use only $3rn$ space. But unfortunetly computation of canonical decomposition and canonical rank can be impossible numerically (known results from tensors theory over finite fields says that computation of canonical rank is NP problem).
I'll end this lecture by one interesting fact about border rank.

Statement 4. *Suppose that t has $brk < rk$. And $t_n \to t$ s. t. $rk(t_n) = brk(t) = r$ and $t_n = \sum_i^r a_i^n \otimes b_i^n \otimes c_i^n$ - any decomposition. Then exist indexes $i_1, i_2, i_1 \neq i_2$ s. t. $|a_{i_1}^n \otimes b_{i_1}^n \otimes c_{i_1}^n| \to \infty, |a_{i_2}^n \otimes b_{i_2}^n \otimes c_{i_2}^n| \to \infty$ for any norm.*

Proof. Suppose that for every i norms of $a_i^n \otimes b_i^n \otimes c_i^n$ are bounded. Then by BolzanoWeierstrass theorem we can extract convergent subsequence. By ex. 5 this subsequence converge to rank one tensor. Suppose that we already extracted subsequences for every i. Then $a_i^n \otimes b_i^n \otimes c_i^n \to a_i \otimes b_i \otimes c_i$ and $t_n = \sum a_i^n \otimes b_i^n \otimes c_i^n \to \sum a_i \otimes b_i \otimes c_i = t$. Hence $rk(t) = brk(t)$ - contradiction. Suppose that we have only one such index. But then $|t_n| \to \infty$ what impossible. □

Excercise 8. *Prove that* $\max_t brk(t) \geq \max(\min(n, mq), \min(nm, q), \min(nq, m))$.