Tasks for the course "Numerical methods in higher dimensions"

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1 List of problems

1. Prove that $\sin(x_1 + \ldots + x_d)$ can be represented with the canonical rank d
2. The TT-SVD and stability estimate. For a given tensor $A(i_1, \ldots, i_d)$ the TT-SVD algorithm proceeds as follows: Compute $A(i_1, \ldots, i_d) = \sum_{\alpha_1} U_1(i_1, \alpha_1) V(\alpha_1 i_2, i_3 \ldots i_d)$ (by the SVD) with $U_1(i_1)$ orthogonal; then separate $(\alpha_1 i_2)$ by the SVD again:
$$V(\alpha_1 i_2, i_3 \ldots i_d) \approx \sum_{\alpha_2} U_2(\alpha_1, i_2, \alpha_2) V_2(\alpha_2 i_3, \ldots, i_d),$$
and the process continues (separate $\alpha_2 i_3$ and so on). In the end, we have the TT-decomposition.

   Prove the stability estimate of the TT-SVD algorithm:
$$||A - TT|| \leq \sqrt{\sum_{k=1}^{d-1} \varepsilon_k^2},$$
where $A_k = R_k + E_k$, $||E_k|| = \varepsilon_k$, and the rank of R_k is r_k (and A_k is the k-th unfolding of A)
3. Show how Kronecker-product approximation of matrices can be done by the SVD, i.e. how to reduce the problem of minimizing
$$||A - \sum_{k=1}^{r} U_k \otimes V_k||$$
to the singular value decomposition (and $\otimes$ is the Kronecker product of matrices)
4. Estimate the complexity of the scalar product computation of two TT-tensors:
$$\langle A, B \rangle = \sum_{i_1, \ldots, i_d} A(i_1, \ldots, i_d) B(i_1, \ldots, i_d),$$
with both tensors in the TT-format.
5. Prove that the two-particle tensor can be approximated by a tensor of canonical rank $\mathcal{O}(d)$, where the "two-term" tensor is defined as
$$T = \sum_{i \leq j} \sigma_{ij} I \otimes \ldots \otimes A_i \otimes \ldots \otimes A_j \otimes I \otimes \ldots \otimes I,$$
i.e. the sum of operators acting only in two modes (for the Laplacian-type operator only one-particle terms are present).