Mathematics of Finance

Finance, like so many other fields, is rife with jargon and specialized terminology. As is often the case, if we strip away the linguistic aspect, we find problems that are familiar to many mathematicians. Many of the problems we attempt to solve in finance involve solving for the initial value where the dynamics governing movement and the terminal boundary value are known. In finance this is the European style option pricing problem. The American Option pricing problem replaces the fixed boundary problem with a free boundary value problem. It also is easily represented as a optimal stopping time problem. The well known Black-Scholes solution, a breakthrough when first reported in 1973, is now taught in undergraduate finance classes.

To derive the Black-Scholes equation, we consider stock (just as easily modeled as a particle on the (non-negative) real line) whose value at time 0, S_0, is known and whose movement is governed by the dynamics

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW,$$

where W is standard Brownian Motion. We are interested in the initial value of a (option price) function $f(S,t)$ where we know the value at time $t = T$ to be $\max(S - K, 0)$ for some constant K. By Ito's lemma, we know:

$$df(s,t) = f_s\, ds + f_t\, dt + \frac{1}{2} f_{ss}\, (ds)^2.$$

The only term with embedded stochastics is the $f_s\, ds$ term. Consider the (instantaneous) dynamics of a portfolio consisting of one unit of the option and short $f_s(s_0, t_0)$ units of S. The (instantaneous) dynamics of this portfolio is given by

$$d\left[f - f_{s_0} S\right] = f_t\, dt + \frac{1}{2} f_{ss}\, (ds)^2.$$

This portfolio (instantaneously) has no stochastic element. At this point in the analysis, finance comes to play. To avoid arbitrage, that is the ability to make riskless profits on zero investment, it is necessary that the prices of any deterministic portfolio (one whose growth rate is non-stochastic) be equal to a universal (for that currency) risk free rate r_{t_0}. The consequence of this is that

$$\begin{aligned} d\left[f - f_{s_0} S\right] &= f_t\, dt + \frac{1}{2} f_{ss}\, (ds)^2 \\ &= r\left[f - f_{s_0} S\right]. \end{aligned}$$

A few substitutions and a little algebra later, one gets the standard PDE for the heat equation. What is very interesting is that the drift μ of the stock has been replaced by r, the risk free rate. Moreover, one eliminated the stochastic element by taking an appropriate position in the underlying security S. Much mathematics will be generated by these observations.

However, if this were all there was to mathematical finance, it would not be of much interest to mathematicians. Fortunately for us, there is much, much more. Listed below are just a few of the issues that generate interest and real complexity:

1. The risk free rate is not constant, but rather governed by stochastic dynamics.

2. The volatility σ is also governed by stochastic dynamics.

3. The risk free rate is currency specific.

4. FX rates (for translating value in one currency to a second) are governed by stochastic laws.

5. The dynamics of many of the interesting derivatives for which we are trying to find the initial value are governed by the joint dynamics of multiple other securities (Not just one stock, but sometimes as many as 100 other securities.)

6. The financial markets are not governed by simple diffusion equations, rather one can expect jump diffusion process with stochastic volatility (is governed by some process.)

7. The parameters in many of these processes are not known, but must be inferred by solving for the parameters from certain observable other derivatives. (Inverse problem).

8. The above analysis is predicated upon overly simplistic assumptions of continuous trading, no impact upon prices resulting from trading (changing portfolios), no additional costs imposed by trading, and the ability to create a portfolio (linear combination from a set of variables) to (instantaneously) eliminate stochastic components.

Each of the talks given by the Morgan Stanley researchers will touch on one or several of the above complexities.