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ON THE CONEIGENVALUES AND SINGULAR VALUES OF A COMPLEX SQUARE MATRIX

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It is shown that the coneigenvalues of a matrix, when properly defined (in a way different from the one commonly used in the literature), obey relations similar to the classical inequalities between the (ordinary) eigenvalues and singular values. Several interesting spectral properties of conjugate-normal matrices are indicated. This matrix class plays the same role in the theory of unitary congruences as the class of normal matrices plays in the theory of unitary similarities. Bibliography: 5 titles.

1. Introduction

Matrices $A, B \in M_n(\mathbf{C})$ are said to be consimilar if $A = SB\overline{S}^{-1}$ for a nonsingular matrix $S \in M_n(\mathbf{C})$. As usual, the bar over the symbol of a matrix means elementwise conjugation. Unitary congruence is an important particular case of consimilarity where $S = U$ is a unitary matrix and $A = UBU^T$.

In accordance with the definition given in [1, Sec. 4.6] (slightly modified to suit our purposes), a scalar $\mu \in \mathbf{C}$ and a nonzero vector $x \in \mathbf{C}^n$ are called a coneigenvalue and a coneigenvector (associated with μ) of a matrix A, respectively, if

$$Ax = \mu\overline{x}. \tag{1}$$

It can be shown (see [1, Sec. 4.6]) that μ is a coneigenvalue of A if and only if $|\mu|^2$ is an (ordinary) eigenvalue of $\overline{A}A$. Therefore, if $\overline{A}A$ has no real nonnegative eigenvalues, then A has no coneigenvalues. If μ is a coneigenvalue, then, for all $\theta \in \mathbf{R}$, $e^{i\theta}\mu$ also is a coneigenvalue. Hence if A has a coneigenvalue, then it has infinitely many of them. By contrast, a matrix of order n always has exactly n (ordinary) eigenvalues if their multiplicities are counted. It follows that the set of coneigenvalues is inconvenient to work with.

In Sec. 2 of this paper, we suggest a different definition of coneigenvalues. In accordance with this definition, any matrix of order n has exactly n coneigenvalues (with account for their multiplicities). It turns out that certain relations between the (ordinary) eigenvalues and matrix norms and also between the eigenvalues and the singular values have counterparts for the coneigenvalues.

Some classical inequalities, such as the Schur inequality or the additive Weyl inequalities, become equalities for a normal matrix A. In Sec. 3, we show that in the case of coneigenvalues, similar equalities hold for the conjugate-normal matrices. In the theory of unitary congruences, this matrix class plays a role similar to that of the normal matrices in the theory of unitary similarities. Other analogous properties of matrices in these two classes are also indicated.

2. Inequalities between the coneigenvalues and the singular values

Given a matrix $A \in M_n(\mathbf{C})$, we associate with it the matrices

$$A_L = \overline{A}A \tag{2}$$

and

$$A_R = A\overline{A}. \tag{3}$$

Although, in general, the products AB and BA need not be similar, the matrices A_L and A_R always are similar (see [1, Sec. 4.6, Problem 9]). Therefore, in the subsequent discussion of spectral properties of these matrices, it will be sufficient to consider only one of them, say, A_L.

The spectrum of A_L has the following remarkable properties.

1. It is symmetric about the real axis. Moreover, the eigenvalues λ and $\overline{\lambda}$ are of the same multiplicity.
2. The negative real eigenvalues of A_L (if any) are necessarily of even algebraic multiplicity.

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For the proofs of these properties, we refer the reader to [1, Sec. 4.6, Problems 5–7].

Let

$$\lambda(A_L) = \{\lambda_1, \dots, \lambda_n\} \tag{4}$$

be the spectrum of A_L and let

$$\rho(A) = \max\{|\lambda|, \lambda \in \lambda(A)\}$$

denote the spectral radius of A.

Definition 1. *The coneigenvalues of A are the n scalars $\mu_1, \dots, \mu_n$ defined as follows:*

- *if $\lambda_i \in \lambda(A_L)$ does not lie on the negative real semi-axis, then the corresponding coneigenvalue μ_i is defined as the square root of λ_i with nonnegative real part, and the multiplicity of μ_i is that of λ_i, i.e.,*

$$\mu_i = \lambda_i^{\frac{1}{2}}, \quad \operatorname{Re}\mu_i \geq 0; \tag{5}$$

- *with a real negative eigenvalue $\lambda_i \in \lambda(A_L)$ we associate two conjugate purely imaginary coneigenvalues*

$$\mu_i = \pm\lambda_i^{\frac{1}{2}}, \tag{6}$$

the multiplicity of each of them being half the multiplicity of λ_i.

The set

$$c\lambda(A) = \{\mu_1, \dots, \mu_n\} \tag{7}$$

is called the conspectrum of A.

The coneigenvalues of a matrix A allow for another interpretation. Define the matrix

$$\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}. \tag{8}$$

Proposition 1. *Let $\mu_1, \dots, \mu_n$ be the coneigenvalues of an $n \times n$ matrix A. Then*

$$\lambda(\widehat{A}) = \{\mu_1, \dots, \mu_n, -\mu_1, \dots, -\mu_n\}. \tag{9}$$

Proof. The assertion desired follows from two observations. First, we have $\widehat{A}^2 = A_R \oplus A_L$, which implies that any eigenvalue of $\widehat{A}$ is a square root of an eigenvalue of A_L. Second, the characteristic polynomial $\varphi(\lambda)$ of $\widehat{A}$ is given by

$$\varphi(\lambda) = \det(\lambda I_{2n} - \widehat{A}) = \det(\lambda^2 I_n - A_L) = \det(\lambda^2 I_n - A_R).$$

Thus, if λ is an eigenvalue of $\widehat{A}$, then $-\lambda$ also is an eigenvalue of $\widehat{A}$, and both of them have the same multiplicity. □

For the rest of this section, we adopt the following conventions.

1. The singular values of A are arranged in nonincreasing order, i.e.,

$$\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_n(A), \tag{10}$$

whence

$$\sigma_{\max}(A) = \sigma_1(A) = ||A||_2. \tag{11}$$

2. The coneigenvalues of A are numbered in nonincreasing order of their absolute values, i.e.,

$$|\mu_1(A)| \geq |\mu_2(A)| \geq \dots \geq |\mu_n(A)|. \tag{12}$$

3. The same conventions apply to the singular values and eigenvalues of $\widehat{A}$, i.e.,

$$\sigma_1(\widehat{A}) \geq \sigma_2(\widehat{A}) \geq \dots \geq \sigma_{2n}(\widehat{A}), \tag{13}$$

$$|\lambda_1(\widehat{A})| \geq |\lambda_2(\widehat{A})| \geq \dots \geq |\lambda_{2n}(\widehat{A})|. \tag{14}$$

In view of (9), we have

$$|\lambda_{2i-1}(\widehat{A})| = |\lambda_{2i}(\widehat{A})| = |\mu_i(A)|, \quad i = 1, 2, \ldots, n. \tag{15}$$

Note that $\widehat{A}$ has the same singular values as $A \oplus \overline{A}$, and $\overline{A}$ has the same singular values as A. Consequently, the singular values of $\widehat{A}$ are those of A repeated twice. Thus,

$$\sigma_{2i-1}(\widehat{A}) = \sigma_{2i}(\widehat{A}) = \sigma_i(A), \quad i = 1, 2, \ldots, n. \tag{16}$$

Finally, we define the conspectral radius of A as follows:

$$c\rho(A) = |\mu_1(A)|. \tag{17}$$

Proposition 2. *Let $||\cdot||$ be an absolute matrix norm. Then*

$$c\rho(A) \le ||A||. \tag{18}$$

Proof. We have

$$c\rho^2(A) = |\mu_1(A)|^2 = \rho(\overline{A}A) \le ||\overline{A}A|| \le ||\overline{A}||\;||A|| = ||A||^2.$$

□

The spectral norm is not absolute. However, inequality (18) holds true for the spectral norm as well.

Proposition 3. *The following inequality is valid:*

$$c\rho(A) \le ||A||_2. \tag{19}$$

Proof. For any submultiplicative matrix norm $||\cdot||$, we have $\rho(\widehat{A}) \le ||\widehat{A}||$, implying that

$$|\lambda_1(\widehat{A})| \le \sigma_1(\widehat{A}).$$

In view of (11) and (15)–(17), this is the desired inequality (19) in disguised form. □

Remark. In a personal communication, R. Horn indicated to the author that Propositions 2 and 3 can be united and strengthened under the assumption that for the matrix norm used, $||A|| = ||\overline{A}||$. Indeed, in this more general case, the proof of Proposition 2 remains the same. In particular, not only the spectral norm but all the unitarily invariant norms are covered.

Proposition 4. *The coneigenvalues satisfy the inequality*

$$\sum_{i=1}^{n} |\mu_i(A)|^2 \le ||A||_F^2. \tag{20}$$

Proof. In application to $\widehat{A}$, the well-known Schur inequality yields

$$\sum_{i=1}^{2n} |\lambda_i(\widehat{A})|^2 \le ||\widehat{A}||_F^2. \tag{21}$$

Obviously,

$$||\widehat{A}||_F^2 = 2||A||_F^2,$$

which, together with (15), shows that (21) is equivalent to (20). □

Since

$$||A||_F^2 = \sum_{i=1}^{n} \sigma_i^2(A),$$

relation (20) can be regarded as an inequality between the coneigenvalues of A and its singular values. From this point of view, the following theorem is an extension of Proposition 4.

Theorem 1. *For $1 \le m \le n$ and an arbitrary real nonnegative δ,*

$$\sum_{i=1}^{m} |\mu_i(A)|^\delta \le \sum_{i=1}^{m} \sigma_i^\delta(A). \tag{22}$$

Proof. By applying the additive Weyl inequalities (see [2, Sec. II.4.2]) to $\widehat{A}$, we obtain

$$\sum_{i=1}^{\ell} |\lambda_i(\widehat{A})|^\delta \le \sum_{i=1}^{\ell} \sigma_i^\delta(\widehat{A}), \quad 1 \le \ell \le 2n. \tag{23}$$

Setting $\ell = 2m$ ($1 \le m \le n$) in (23) and taking into account (15) and (16), we arrive at (22). □

We conclude this section with the following simple but useful result.

Proposition 5. *Let A be a block triangular matrix of the form*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

Then

$$c\lambda(A) = c\lambda(A_{11}) \cup c\lambda(A_{22}). \tag{24}$$

Of course, (24) also holds for a lower block triangular matrix. Moreover, analogous equalities are valid not only for 2×2 block triangular matrices but for all block orders.

3. Conjugate-normal matrices

The role of normal matrices in the theory of unitary similarities is well known. It is related to the fact that the normal matrices are exactly the matrices that can be brought to the simplest (diagonal) form by unitary similarity transformations. The conjugate-normal matrices (c.n. matrices) play a similar role in the theory of unitary congruences.

Definition 2. *A matrix $A \in M_n(\mathbf{C})$ is said to be conjugate-normal if*

$$AA^* = \overline{A^*A}. \tag{25}$$

Seemingly, the class of c.n. matrices was first introduced in [3]. A canonical form for c.n. matrices with respect to unitary congruence transformations was also found in [3].

Theorem 2. *Any conjugate-normal matrix $A \in M_n(\mathbf{C})$ can be brought by a proper unitary congruence transformation to a block diagonal matrix with diagonal blocks of orders 1 and 2. The 1×1 blocks are the nonnegative coneigenvalues of A. Each 2×2 block corresponds to a pair of complex conjugate coneigenvalues $\mu_j = \rho_j e^{i\theta_j}, \overline{\mu}_j$ and is of the form*

$$\begin{bmatrix} 0 & \rho_j \\ \rho_j e^{-i2\theta_j} & 0 \end{bmatrix} \tag{26}$$

or

$$\begin{bmatrix} 0 & \mu_j \\ \overline{\mu}_j & 0 \end{bmatrix}. \tag{27}$$

The block diagonal matrix described in Theorem 2 is called the canonical form of the c.n. matrix A. The form (26) of its 2×2 blocks was used in [3], whereas the alternative form (27) was given in [4].

Complex symmetric, skew-symmetric, and unitary matrices are special cases of c.n. matrices. From the classical Takagi theorem (see [1, Sec. 4.4]) it follows that the coneigenvalues of a symmetric matrix are identical to its singular values. The coneigenvalues of a unitary matrix U, being the square roots of the (ordinary) eigenvalues of the unitary matrix $\overline{U}U$, have unit absolute values; on the other hand, all the singular values of U are equal to one. This relation between the coneigenvalues and the singular values holds for the entire class of c.n. matrices.

Proposition 6. *The singular values of a conjugate-normal matrix A are the absolute values of its coneigenvalues.*

Proof. The relation desired is readily obtained by inspecting the canonical form of A. Indeed, the nonnegative coneigenvalues (i.e., the 1×1 blocks in the canonical form) are singular values of A. On the other hand, the singular spectrum of matrix (27) is the scalar $|\mu_j|$ repeated twice. $\square$

Corollary 1. *For a conjugate-normal matrix A, inequalities* (19), (20), *and* (22) *hold with equality.*

Remark. As was shown in [5], the equality in (20) can be attained only for a conjugate-normal matrix A.

In conclusion, we discuss another interesting property of the c.n. matrices. First we recall that the Toeplitz (or Cartesian) decomposition of a complex square matrix A is defined as the representation

$$A = B + C, \qquad B = B^*, \quad C = -C^*. \tag{28}$$

The matrices B and C, called the real and imaginary parts of A, respectively, are uniquely determined by the equalities

$$B = \frac{1}{2}(A + A^*), \qquad C = \frac{1}{2}(A - A^*).$$

The usefulness of the Toeplitz decomposition is related to the fact that it is respected by unitary similarity transformations in the following sense: for a unitary matrix U, the matrices U^*BU and U^*CU are the real and imaginary parts of U^*AU, respectively; in addition, under the transformation with the matrix U, all the three matrices A, B, and C preserve their eigenvalues.

The representation

$$A = S + K \tag{29}$$

of a matrix A, where

$$S = \frac{1}{2}(A + A^T) \quad \text{and} \quad K = \frac{1}{2}(A - A^T) \tag{30}$$

are a symmetric and a skew-symmetric matrices, called the symmetric and skew-symmetric parts of A, respectively, will be referred to as its SSS (meaning Symmetric-Skew-Symmetric) decomposition. Decomposition (29), (30) is the counterpart of the Toeplitz decomposition for the theory of unitary congruences.

Decomposition (29), (30) is respected by unitary congruence transformations in the sense that for a unitary U, the matrices U^TSU and U^TKU are the symmetric and skew-symmetric parts of the matrix U^TAU, respectively. Moreover, the coneigenvalues of the three matrices A, S, and K are preserved under unitary congruence transformations.

Theorem 3. *Let A be a conjugate-normal matrix with SSS decomposition* (29), (30). *Then the coneigenvalues of the matrices S and K are the real and imaginary parts, respectively, of the coneigenvalues of A.*

Proof. This can readily be seen by inspecting the canonical form of A. If μ is a 1×1 block in the canonical form, then, obviously, its SSS decomposition is

$$\mu = \mu + 0.$$

If $\mu_j = x_j + iy_j$ is a complex coneigenvalue of A, then the SSS decomposition of matrix (27) is of the form

$$S_j + K_j,$$

where

$$S_j = \begin{bmatrix} 0 & x_j \\ x_j & 0 \end{bmatrix} \tag{31}$$

and

$$K_j = \begin{bmatrix} 0 & iy_j \\ -iy_j & 0 \end{bmatrix}. \tag{32}$$

The conspectrum of matrix (31) is the scalar x_j repeated twice, whereas matrix (32) has the coneigenvalues iy_j and $-iy_j$. $\square$

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