Solution of linear systems with square sparse matrices

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Outline

- **Systems with sparse matrices**
- **Direct methods**
- **Iterative methods**
- **Concluding remarks**

Definitions

Let A be a square matrix of order n, $detA \neq 0$:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

How can we solve a linear system with A and a right hand side $b \in \Re^n$, i.e. find such $x \in \Re^n$ that

$$Ax = b?$$

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Example: finite differences for $-u'' = f$ **in** $(0,1)$**,** $u(0) = u(1) = 0$**.**

$$A_1 = h^{-2} \begin{pmatrix} 2 & -1 & 0 & \dots \\ -1 & 2 & -1 & 0 \\ & \ddots & \ddots & \ddots \\ 0 & & & \\ \dots & 0 & -1 & 2 \end{pmatrix}$$

Meshes and spectral properties of matrices

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Finite differences for $-\Delta u = f$ in $\Omega = (0,1)^2$, $u = 0$ on $\partial\Omega$:

$$A_{12} = A_1 \otimes I_2 + I_1 \otimes A_2$$

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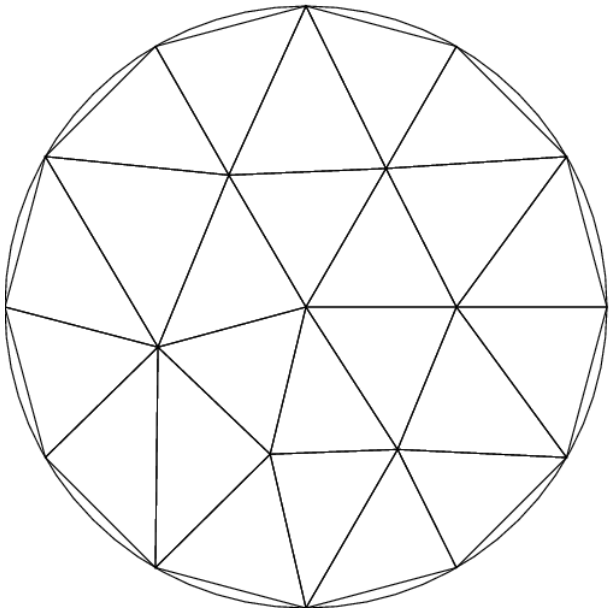

quasiuniform mesh with mesh size h

$$\lambda_{\min} = O(h^2),\ \lambda_{\max} = O(1)\ \rightarrow\ cond\, A = O(h^{-2})$$

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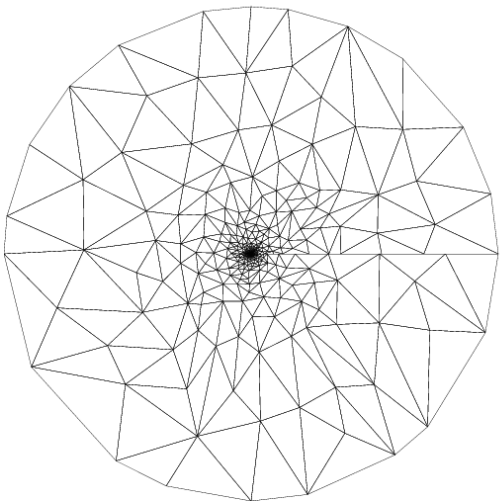

shape regular mesh with mesh size $[h_{\min}, h_{\max}]$

but may be

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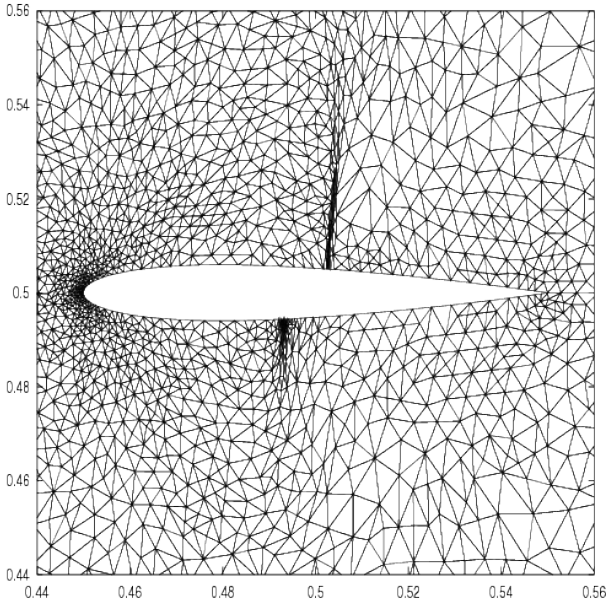

adaptive anisotropic mesh

$$\lambda_{\min} = O((\alpha h)^2_{\min}),\ \lambda_{\max} = O(1)$$

$$cond\, A = O((\alpha h)^{-2}_{\min})$$

Classes of solvers

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1. **Find permutations and factorize:**

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  - **May be inefficient for systems of PDE's**
- **Implementations: UMFPACK, MUMPS, WSMP, PARDISO, SuperLU**

Sparse factorization

2D Poisson problem

$$A_{12}x = b$$

$$A_{12} = A_1 \otimes I_2 + I_1 \otimes A_2$$

$$nz_{LU} = O(n \ln n), \quad Flops = O(n^{\alpha}), \quad 1.5 \leq \alpha \leq 1.6$$

n_A	nz_A	MB	nz_{LU}	t_{factor}	t_{solve}
640000	**3196800**	**485**	$\mathbf{42 \cdot 10^6}$	**82**	**0.97**

Sparse factorization

3D Poisson problem

$$A_{123}x = b$$

$$A_{123} = A_1 \otimes I_2 \otimes I_3 + I_1 \otimes A_2 \otimes I_3 + I_1 \otimes I_2 \otimes A_3$$

$$nz_{LU} = O(n^{\alpha}), \quad 1.5 \leq \alpha \leq 1.6, \quad Flops = O(n^{\beta}), \quad 2.2 \leq \beta \leq 2.3$$

n_A	nz_A	MB	nz_{LU}	t_{factor}	t_{solve}
125000	**860000**	**926**	$\mathbf{82 \cdot 10^6}$	**1107**	**1.57**

Fast direct solver: Devide and conquer

Matrices from a special class

Two stages:

- **Initialization**
- **Solution**

Yuri Kuznetsov '84

$$A_{12} = A_1 \otimes I_2 + I_1 \otimes A_2 + cI_1 \otimes I_2$$

rectangular non-uniform mesh

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Systems of order 10^8 are solved at laptops within 1 min!

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Advantages:

- **Efficient**
- **Robust**
- **Parallelizable**

Abstract iterative solver

$$Ax = b$$

- **Choose an initial guess** x_0

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- $\dots$ **depend on the method**

Classical methods

Richardson's iterations:

$$x_k = x_{k-1} + \alpha(b - Ax_{k-1})$$

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In case of A_1 methods provide slow convergence, but fast frequencies are dumped quickly

$$\|e_k\| \leq (1 - ch^2)\|e_{k-1}\|, \quad e_k = x - x_k$$

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In case of A_1 methods provide better convergence if ω is optimal:

$$\|e_k\| \leq (1 - ch)\|e_{k-1}\|, \quad e_k = x - x_k$$

Krylov subspace methods

- $x_m - x_0 \in \mathcal{K}_m$, $\mathcal{K}_m = span\{r_0, Ar_0, \ldots, A^{m-1}r_0\}$, $r_0 = b - Ax_0$

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- **Examples: CG (for $A = A^T > 0$), Lanczos (for $A = A^T$), GMRES**
- **Objectives: automatic choice of parameters and fast convergence**
- **Convergence depends on spectral bounds. In case of A_1 and CG:**

$$\|e_k\|_A \leq 2(1 - ch)^k \|e_0\|_A$$

Preconditioning

$$Ax = b \Rightarrow M^{-1}Ax = M^{-1}b$$

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- **Krylov subspace methods may be preconditioned**

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- **Symmetric SOR** $M = (D - \omega E)D^{-1}(D - \omega F)$

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- **Example: ILU (I.Kaporin,98), Poisson equation in $(0,1)^2$**

mesh	n_A	nz_A	nz_{LU}	t_{factor}	t_{solve}	itr_{PCG}
$1t_{2h}$	**66049**	**460289**	$\mathbf{1.1 \cdot 10^6}$	**2.2**	**2.5**	**62**
$1t$	**263169**	**1838081**	$\mathbf{4.4 \cdot 10^6}$	**11.6**	**27.2**	**126**
3c	**229343**	**1605267**	$\mathbf{3.7 \cdot 10^6}$	**11.8**	**39.9**	**242**

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dependence on h: [$1t_{2h}$,1t]
dependence on mesh structure: [1t,3c]

$nz_{LU} \sim n$ **for the same sparsity**

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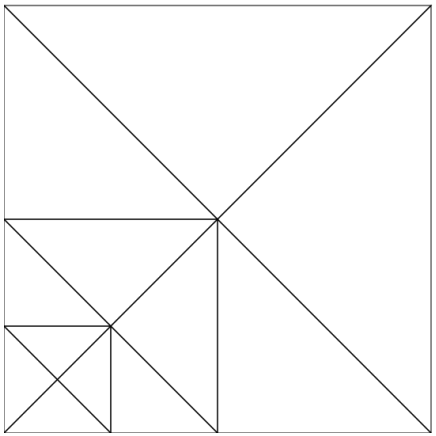
Extensions:



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Multigrid preconditioner

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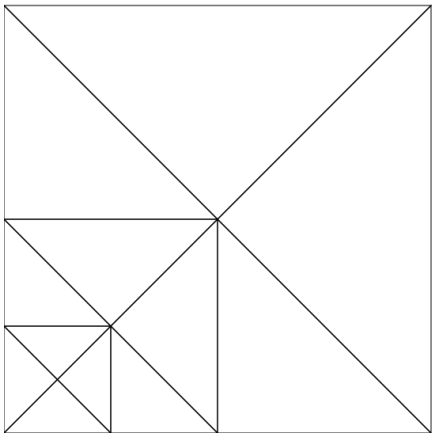

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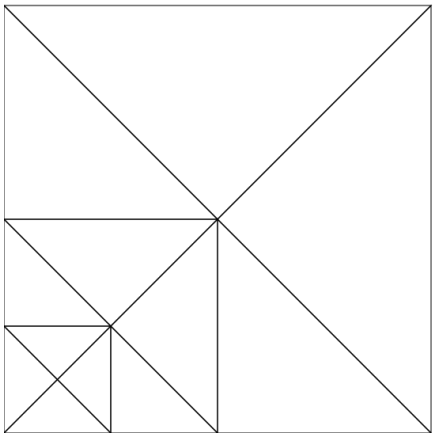

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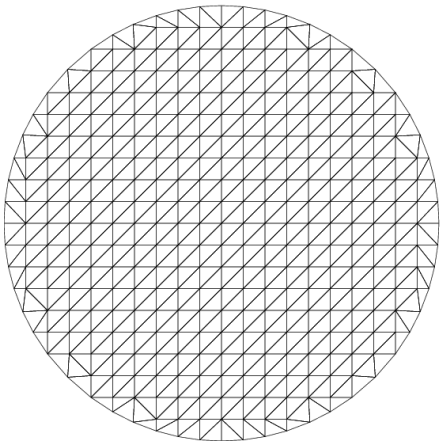
Fictitious domain preconditioner



FEM discretization of Poisson equation with Neumann b.c.

Astrakhancev, Kuznetsov, Matsokin 70's

$$\begin{pmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma}^{(1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_\Gamma \end{pmatrix} = \begin{pmatrix} b_1 \\ b_\Gamma \end{pmatrix}$$

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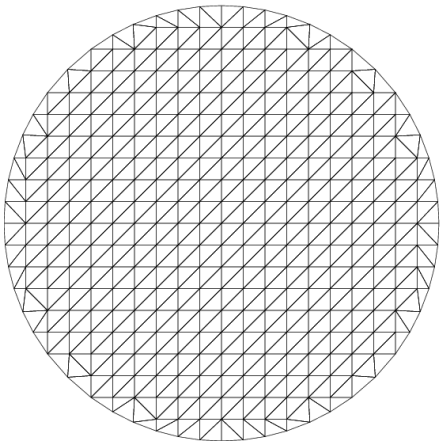

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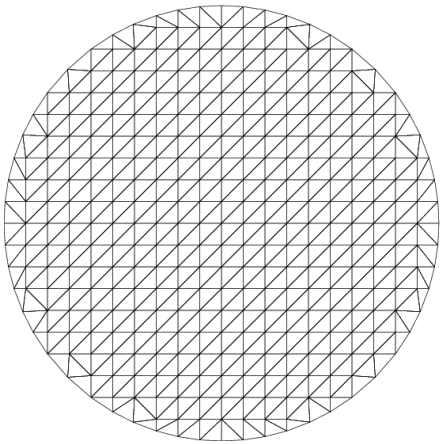

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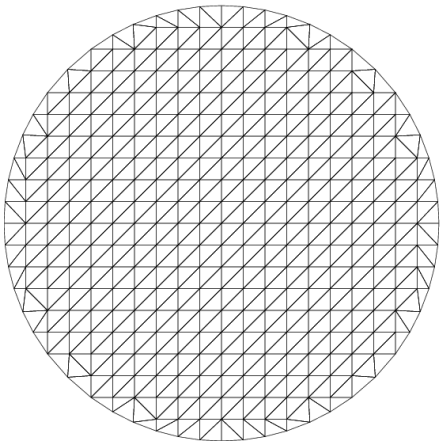

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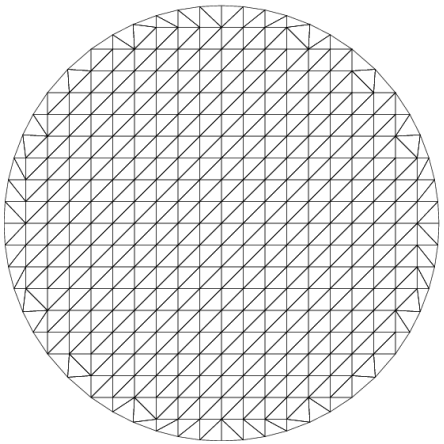

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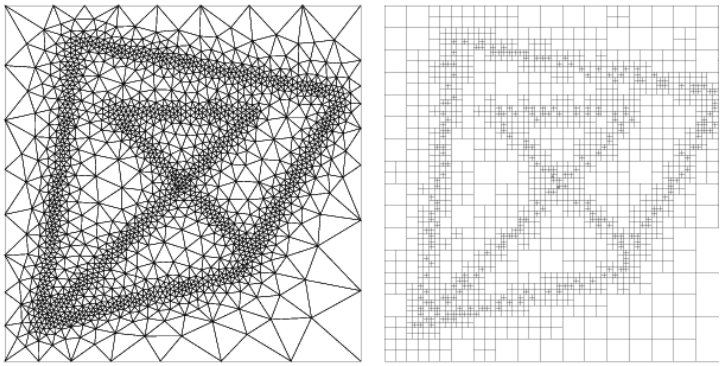
fast convergence, parallel, partial solutions

Fictitious space preconditioner



Nepomnyaschikh ’91

$$M^{-1} = R\tilde{M}^{-1}R^{\star}$$

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- **Fast convergence, parallel, only precondtioners, extension to large m (Mandel '93)**

Conclusions

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