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Effective computation of Chebyshev polynomials for several intervals

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Abstract. A cell decomposition of the space of polynomials $T_n(E,x)$ of least deviation from zero on a system E of several closed intervals of the real axis is discussed. An effective method for calculating the T_n in each cell making use of automorphic functions is put forward.

Bibliography: 29 titles.

§ 1. Introduction

Let E be a collection of several intervals of the real axis with extreme left and extreme right points -1 and $+1$, respectively. We consider the classical problem of the polynomial of least deviation on E: *in the class of polynomials*

$$P_n(x) = x^n + \cdots \tag{1}$$

of degree n with real coefficients find the polynomial of minimum $C(E)$-norm

$$L_n = \inf_{P_n} \max_{x\in E} |P_n(x)|. \tag{2}$$

We call the solution of the problem (1), (2), divided by the deviation of L_n from zero, the *Chebyshev polynomial* $T_n(E,x)$ of the set E.

The problem of least deviation goes back to Chebyshev, who solved it in the case of $E = [-1,1]$. In many special cases of the form $E = [-1,a] \cup [b,1]$, $-1 < a < b < 1$, the solution of the problem was obtained by Akhiezer [1], but in fact, can also be found in Zolotarëv's papers [2] and [3] (see also [4]), who did not himself discuss the problem. The solutions in these papers were constructive, in terms of elliptic functions. In the case when E is a union of several intervals some problems of least deviation were solved by Lebedev [5]. In a recent paper [6] Peherstorfer and Schiefermayr develop a method for computing the polynomials of least deviation on several intervals, which however is efficient only for small degree n. Akhiezer [7] suggested Schottky automorphic functions for a parametric representation of solutions in the general case, but this approach (which was used

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for extremal rational functions, see [8]) so far has not been sufficiently developed to produce quantitative results. It is this approach that we concentrate on in our paper.

We now formulate the problem that we are solving; it should be pointed out from the outset that this is not the problem of least deviation (1), (2). We embed the space of Chebyshev polynomials $\bigcup_E T_n(E,\,\cdot\,)$ of degree n in the space of polynomials of degree at most n, which we identify with $\mathbb{R}^{n+1}$. Using Akhiezer's 'comb' domains [9], [10] one can show that the space of Chebyshev polynomials is an unbounded closed subset of $\mathbb{R}^{n+1}$ diffeomorphic to an $(n-1)$-dimensional coordinate sector. This sector $[0,\infty)^{n-1}$ has a natural cell decomposition into sectors of smaller dimension lying in various planes spanned by coordinate axes. The corresponding decomposition of the space of Chebyshev polynomials has the following form:

$$\bigcup_E T_n(E,\,\cdot\,)=\bigcup_{g=0}^{n-1}\bigcup_{m_0,\dots,m_g}\mathbb{T}(m_0,m_1,\dots,m_g), \tag{3}$$

where the positive integers $m_0,m_1,\dots,m_g$, of total sum n, describe the topological structure of the graph of T_n, and $\mathbb{T}(m_0,m_1,\dots,m_g)$ is a smoothly embedded g-dimensional cell. Thus, to each Chebyshev polynomial we can assign a collection of parameters peculiar to it: discrete parameters $\{m_k\}_{k=0}^g$ that specify the cell $\mathbb{T}$ and continuous parameters ν_k, $0<\nu_k<\infty$, $k=1,\dots,g$, that are global coordinate functions in that cell. *The computation of the Chebyshev polynomials $T_n(\,\cdot\,,x)$ in terms of the values of these parameters is the subject of the present paper.*

Unfortunately, we do not know the map $E\to\bigl[g;\{m_k\}_{k=0}^g;\{\nu_k\}_{k=1}^g\bigr]$, which could be helpful in the solution of the problem of least deviation (1), (2). More than that, this map is unstable. By contrast, the map $\bigl[g;\{m_k\}_{k=0}^g;\{\nu_k\}_{k=1}^g\bigr]\to\{E\}$ associating with a fixed T_n all corresponding least deviation sets E (such that $T_n=T_n(E,\,\cdot\,)$) has a simple structure and we describe it in Theorem 2.

A construction going back to Akhiezer enables one to embed each g-dimensional cell $\mathbb{T}(m_0,\dots,m_g)$ in some moduli space $\mathcal{H}(\mathbb{R};g,1)$ of twice the dimension. The moduli space $\mathcal{H}$ consists of hyperelliptic curves M of genus g with real ramification points and one distinguished real point, and the embedded cell $\mathbb{T}$ is described by Abel's equations

$$\int_{\widetilde{\mathbf{B}}_s} d\eta=2\pi i\,\frac{m_s}{n}\,,\qquad s=0,1,2,\dots,g, \tag{4}$$

discussed in § 2.1. For an effective computation of Chebyshev polynomials it suffices to find a representation of Riemann surfaces M making up the moduli space (a model) that could be convenient in the solution of two problems:

(A) the effective resolution of the constraints (4);
(B) the effective recovery of a polynomial $T_n(\,\cdot\,,x)$ from the curve M representing it.

As long ago as 1928, Akhiezer — in connection with another problem of least deviation — proposed the use of the Schottky uniformization of M for the solution of the second problem. The novel feature of our approach is that we solve problem (A) in the same framework.

The *scheme of the computation of Chebyshev polynomials* is as follows: from an arbitrary point in the moduli space, using variational formulae for the left-hand sides of Abel's equations in the moduli space, we perform the gradient descent onto the cell $\mathbb{T} \subset \mathcal{H}(\mathbb{R}; g, 1)$. The Chebyshev polynomial $T_n \in \mathbb{T}$ is calculated by explicit formulae in terms of the Riemann surface $M \in \mathcal{H}(\mathbb{R}; g, 1)$ corresponding to it under the embedding $\mathbb{T} \to \mathcal{H}(\mathbb{R}; g, 1)$.

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§ 2. Structure of Chebyshev polynomials

The qualitative structure of the graph of a Chebyshev polynomial T_n (see Fig. 7) can be understood on the basis of Chebyshev's alternance theorem. The description of the graph is as follows.

Theorem 1. *A polynomial $P_n(x)$ of degree n with positive leading coefficient is the Chebyshev polynomial of some set E if and only if the interval $I := (-1, 1)$ contains n disjoint subintervals $I_1, \dots, I_n$ such that two of them adjoin the end-points of I and $P_n \colon I_s \to I$ is a homeomorphism for each $s = 1, \dots, n$.*

Proof. (1°) Let $T_n(E, x)$ be the Chebyshev polynomial of E and $e_0, e_1, \dots, e_n$ its alternance points. Each interval (e_{s-1}, e_s), $s = 1, \dots, n$, contains precisely one point in the inverse image $T_n^{-1}(y)$ for each $y \in I$. Let $I_s := \{x \in (e_{s-1}, e_s) : T_n(x) \in I\}$. We consider an arbitrary connected component I_s^0 of I_s. The values of T_n at the end-points of I_s^0 are equal to one in absolute value but must be of opposite signs, for otherwise T_n takes equal values at distinct points in $I_s^0 \subset (e_{s-1}, e_s)$. Hence $I_s^0 = I_s$ and the restriction of T_n to I_s is a homeomorphism onto I. Since ± 1 are the extreme points of E, it follows that $-1 \in \partial I_1$ and $1 \in \partial I_n$.

(2°) Let P_n be a polynomial mapping disjoint intervals $I_1, \dots, I_n$ onto I in a one-to-one manner. The monotonicity of P_n on two successive intervals I_{s-1}, I_s has opposite character. A calculation of the number of the alternance points shows that $P_n = T_n(E, \cdot\,)$ for $E := \bigcup_{s=1}^n \overline{I}_s$.

Corollary. *A polynomial $P_n(x)$ of degree n with positive leading coefficient is a Chebyshev polynomial (of some set E) if and only if*

$$\{\pm 1\} \subset P_n^{-1}(\{\pm 1\}) \subset [-1, 1]. \tag{5}$$

Proof. (1°) If P_n is a Chebyshev polynomial, then $P_n^{-1}(\{\pm 1\}) = \bigcup_{s=1}^n \partial I_s$ and the inclusions (5) are beyond any doubt.

(2°) Let $-1 = e_1^* \leqslant e_2^* \leqslant \dots \leqslant e_{2n}^* = 1$ be points making up the inverse image $P_n^{-1}(\{\pm 1\})$, where the number of occurrences of each point is equal to its multiplicity. We claim that the sequence of $s^0(j) := P_n(e_j^*)$, $j = 1, \dots, 2n$, is as follows:

$$s^0 = \{(-1)^n, \dots, +1, +1, -1, -1, +1, +1, -1, -1, +1\}. \tag{6}$$

Consider the 'derivative' sequence $s^1(j) := P_n(e_j^*)P_n(e_{j+1}^*)$, $j = 1, \dots, 2n-1$. The positivity or the negativity of $s^1(j)$ indicates that the interval $[e_j^*, e_{j+1}^*]$ (which may

degenerate into a point) contains a zero of the derivative or a zero of the polynomial itself, respectively. Hence the sequence s^1 contains n negative and $n-1$ positive elements. There is no piece of the form $-1,-1$ in the sequence s^1, for otherwise s^0 must contain either a piece $-1,+1,-1$ or $+1,-1,+1$, which is impossible. Hence $s^1(j)=(-1)^j$, and since the leading coefficient of P_n is positive, it follows that $s^0(2n)=+1$ and the general element of s^0 has the form (6).

The polynomial P_n is monotonic in each interval $I_s:=(e^*_{2s-1},e^*_{2s})$, $s=1,\dots,n$, and takes values ±1 at its end-points. By Theorem 1 this is a Chebyshev polynomial.

It is clear from the proof of Theorem 1 that the same Chebyshev polynomial $T_n(\,\cdot\,,x)$ can be the solution of the least deviation problem (1), (2) for many closed sets E containing more than n points. We list all least deviation sets E in the following theorem.

Theorem 2. *Let T_n be the Chebyshev polynomial for some collection of intervals $T_n^{-1}(I)=:\bigcup_{s=1}^n I_s$. Then $T_n=T_n(E,\,\cdot\,)$ if and only if $E\subset\bigcup_{s=1}^n\overline{I}_s$ and E intersects each of the $n+1$ components of the set $\mathbb{R}\setminus\bigcup_{s=1}^n I_s$.*

Proof. Let $\overline{I}^*_s$ be the closed interval between I_s and I_{s+1} (which may degenerate into a point). The character of the monotonicity of T_n on successive intervals I_s alternates, therefore the polynomial takes the same value (of modulus one) at the end-points of each $\overline{I}^*_s$. For alternance points e_s we have the following possibilities:

$$e_0=-1,\qquad e_s\in\partial\overline{I}^*_s,\quad s=1,\dots,n-1,\qquad e_n=1. \tag{7}$$

(1°) Let E be the least deviation set of $T_n=T_n(E,\,\cdot\,)$. The absolute value of the polynomial on E is at most one, therefore $E\subset\bigcup_{s=1}^n\overline{I}_s$. The intersections of E with each component of the complement of $\bigcup_{s=1}^n I_s$ are non-empty because all alternance points (7) lie in E.

(2°) If a closed set E lies in the union of the closures of the intervals I_s and intersects each component of the complement of the union of these intervals, then we can choose alternance points $e_s\in E$ in accordance with (7) (in more than one way in general). By the alternance theorem $T_n=T_n(E,\,\cdot\,)$.

To parametrize the set of various polynomials $T_n(\,\cdot\,,x)$ one must use quantities strongly attached to the polynomials. For instance, with each Chebyshev polynomial $T_n(E,x)$ one associates ([9], [11], [12]) its *support set* (also called the *n-regular set* or the *maximal set of least deviation*) and also some *ordered partitioning of the degree into a sum of positive integers*.

Definition. The *support set* E^+ of a Chebyshev polynomial T_n is the set

$$E^+:=\bigcup_{s=1}^n\overline{I}_s=T_n^{-1}(\overline{I}); \tag{8}$$

its connected components $E^+_0,E^+_1,\dots,E^+_g$ are indexed from left to right. The *ordered partitioning of the degree n* of the polynomial is the collection of integers m_k equal to the number of subintervals I_s in the components E^+_k:

$$m_0+m_1+\dots+m_g=n. \tag{9}$$

We present below Theorem 1 providing necessary and sufficient conditions for a set

$$E^+ =: [-1,1] \setminus \bigcup_{k=1}^{g} (a_k, b_k), \tag{10}$$

$$(b_0 :=) \ \ -1 < a_1 < b_1 < a_2 < b_2 < \cdots < a_g < b_g < 1 \ \ (=: a_{g+1}), \tag{11}$$

to be the support set of a Chebyshev polynomial with partitioning of the degree (9). Following [13], we associate with each set E^+ of the form (10), (11) the hyperelliptic genus-g curve

$$M(E^+) := \left\{ w^2 = (x^2-1) \prod_{k=1}^{g} (x-a_k)(x-b_k) \right\}, \tag{12}$$

on which we distinguish two canonical homology bases: $\mathbf{A}_1, \ldots, \mathbf{A}_g$; $\mathbf{B}_1, \ldots, \mathbf{B}_g$ and $\widetilde{\mathbf{B}}_1, \ldots, \widetilde{\mathbf{B}}_g$; $\widetilde{\mathbf{A}}_1, \ldots, \widetilde{\mathbf{A}}_g$. We define the homology classes whose representatives are not indicated in Fig. 1 by the formulae

$$\begin{aligned} \widetilde{\mathbf{A}}_s &= \mathbf{A}_1 + \cdots + \mathbf{A}_s, \\ \mathbf{B}_s &= \widetilde{\mathbf{B}}_0 + \cdots + \widetilde{\mathbf{B}}_{s-1}, \end{aligned} \qquad s = 1, \ldots, g. \tag{13}$$

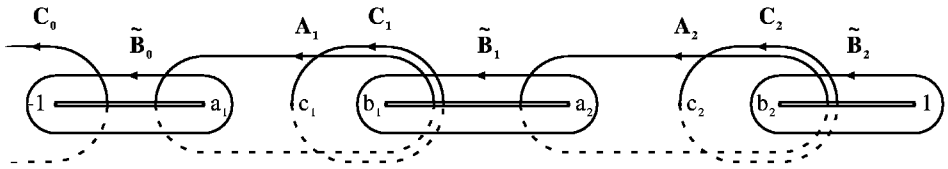


Figure 1. The cycles $\mathbf{A}_s$, $\widetilde{\mathbf{B}}_s$, and the paths $\mathbf{C}_s$ on the Riemann surface $M(E^+)$ of genus $g = 2$

There exists on $M(E^+)$ a unique $\mathbf{A}$-normalized Abelian differential of the third kind $d\eta$ [14] with poles at the points at infinity in M such that $d\eta$ has residue -1 at the pole ∞_+ on the upper sheet (accordingly, the residue at the pole ∞_- on the lower sheet is $+1$). This differential must be real and has precisely one zero in each interval (a_k, b_k), so that

$$d\eta(E^+) = \frac{dx}{w} \prod_{k=1}^{g} (x - c_k), \qquad c_k \in (a_k, b_k). \tag{14}$$

Theorem 3 (Abel's equations) [9], [11], [12], [15]**.** *A set E^+ of the form* (10), (11) *is a support set of a Chebyshev polynomial with partitioning of the degree* (9) *if and only if*

$$\int_{\widetilde{\mathbf{B}}_s} d\eta = 2\pi i \, \frac{m_s}{n}, \qquad s = 0, 1, 2, \ldots, g. \tag{15}$$

A Chebyshev polynomial can be recovered from its support set E^+ satisfying conditions (15) *by the formula*

$$T_n(E^+,x)=\cos\biggl(in\int_{(1,0)}^{(x,w)}d\eta\biggr),\qquad x\in\mathbb{C},\tag{16}$$

where the integral is taken over an arbitrary path in $M(E^+)$ joining $(1,0)$ to (x,w) and does not depend on the choice of one of the two possible values of w.

Remarks. (1) Relations (15) contain only g independent equations because the contour $\widetilde{\mathbf{B}}_0+\dots+\widetilde{\mathbf{B}}_g$ contracts to a pole of $d\eta$.

(2) Riemann's bilinear relations enable one to write (15) also in an equivalent form:

$$\sum_{k=1}^{g}m_k\int_{\widetilde{\mathbf{A}}_k}d\zeta+n\int_{\infty_-}^{\infty_+}d\zeta=0,\tag{17}$$

where $d\zeta$ is an arbitrary differential of the first kind on $M(E^+)$ and integration from ∞_- to ∞_+ proceeds along the path $\mathbf{C}_0$, see Fig. 1. Conditions (17) have the form of Abel's equations defining the divisor of a meromorphic function on $M(E^+)$.

(3) In connection with the problem of least deviation conditions (15) for $g=1$ were first stated by Zolotarëv (see [4]); for $g=2$ and $m_1=m_2=1$ they appeared first — in the form of relation (86) for Schottky functions — in Akhiezer's paper [7]. In the preprint [16] by Kreĭn, Levin, and Nudel'man equations very similar to (15) arise in connection with the problem of the representation of polynomials that are positive on a system of intervals. Equations (15) that we use here were apparently used for the first time by Peherstorfer [11], [15].

(4) Akhiezer [10] (see also [9]) developed a very transparent interpretation of the representation (16). For an arbitrary set E^+ of the form (10) the Schwarz–Christoffel integral

$$ni\int_{-1}^{x}d\eta(E^+),\qquad \operatorname{Im}x>0,$$

maps the upper half-plane (in the upper sheet of $M(E^+)$) onto a 'comb' domain

$$\{z\in\mathbb{C}:0<\operatorname{Im}z,\ 0<\operatorname{Re}z<n\pi\}\setminus\bigcup_{s=1}^{g}[\widetilde{\mu}_s,\widetilde{\mu}_s+i\widetilde{\nu}_s],\tag{18}$$

where the end-points $\widetilde{\mu}_1,\widetilde{\mu}_2,\dots,\widetilde{\mu}_g$, $0<\widetilde{\mu}_1<\widetilde{\mu}_2<\dots<\widetilde{\mu}_g<n\pi$, of the vertical slits and their lengths $\widetilde{\nu}_k$, $0<\widetilde{\nu}_k<\infty$, $k=1,\dots,g$, are described by the formulae

$$\widetilde{\mu}_k:=ni\int_{-1}^{a_k}d\eta,\quad \widetilde{\nu}_k:=ni\int_{a_k}^{c_k}d\eta,\qquad k=1,\dots,g.$$

In 'combs' corresponding to Chebyshev polynomials the $\widetilde{\mu}_k$ are multiples of π. The converse is also true: each 'comb' (18) with slits whose x-coordinates $\widetilde{\mu}_k$ are multiples of π correspond to a Chebyshev polynomial $T_n(\,\cdot\,,x)=(-1)^n\cos\theta(x)$, where $\theta(x)$ is the conformal map of the upper half-plane onto this 'comb' normalized by the condition $\{-1,1,\infty\}\to\{0,n\pi,\infty\}$. This establishes a one-to-one relation between Chebyshev polynomials and 'comb' domains with 'teeth' of lengths $\widetilde{\nu}_k$ at the points $k\pi$, $k=1,\dots,n-1$. (Here we allow slits of length zero, so that the $\widetilde{\nu}_k$ here are in general not the same as in (18).)

Definition. We call the quantities $\widetilde{\nu}_k$, $0 \leqslant \widetilde{\nu}_k < \infty$, $k = 1, \dots, n-1$, the *Akhiezer coordinates* of the Chebyshev polynomial T_n. We denote the *collection of all Chebyshev polynomials* with fixed partitioning of the degree (9) by $\mathbb{T}(m_0, m_1, \dots, m_g)$.

The Akhiezer coordinates enable one to identify the space of Chebyshev polynomials $\bigcup_E T_n(E, \cdot\,)$ and the sector $[0, \infty)^{n-1}$. The sets $\mathbb{T}(m_0, m_1, \dots, m_g)$ make up a natural cell decomposition of this sector into sectors of smaller dimensions: polynomials in $\mathbb{T}(m_0, \dots, m_g)$ correspond to points with Akhiezer coordinates of the following form:

$$(0, \dots, 0, \widetilde{\nu}_{\widetilde{m}_1}, 0, \dots, 0, \widetilde{\nu}_{\widetilde{m}_2}, 0, \dots, 0, \widetilde{\nu}_{\widetilde{m}_g}, 0, \dots, 0),$$
$$\widetilde{m}_k := m_0 + m_1 + \dots + m_{k-1}. \tag{19}$$

The interior of $[0, \infty)^{n-1}$ is the cell $\mathbb{T}(1, 1, \dots, 1)$ containing 'generic' Chebyshev polynomials. Each polynomial $P_n(x)$ with simple real zeros can be moved into interior of the $(n-1)$-dimensional sector by a linear change of the variable x (and multiplication of P_n by a constant). Of greater interest are polynomials lying in cells of lower dimension. For instance, the unique vertex $\mathbb{T}(n)$ of the cell decomposition corresponds to the classical Chebyshev polynomial, the edges $\mathbb{T}(1, n-1), \mathbb{T}(2, n-2)$, $\dots, \mathbb{T}(n-1, 1)$ correspond to the Zolotarëv polynomials of the 1st kind (for the classification of the Zolotarëv polynomials, see [4]); we present the graphs of polynomials in the two-dimensional faces $\mathbb{T}(11, 15, 4), \mathbb{T}(10, 17, 3), \mathbb{T}(13, 2, 15), \dots$ in Fig. 7.

We now embed the Chebyshev polynomials $T_n(E, \cdot\,)$ in the space of polynomials of degree at most n, which we identify with $\mathbb{R}^{n+1}$ by fixing some polynomial basis. The next theorem gives an answer to the following question: which part of the space $\mathbb{R}^{n+1}$ corresponds to the space of Chebyshev polynomials $\bigcup_E T_n(E, \cdot\,)$?

Theorem 4. *The space of Chebyshev polynomials is a subset that is*

(1°) *closed*,
(2°) *unbounded*,
(3°) *lying in a plane of codimension* 2,
(4°) *real-analytically diffeomorphic to an* $(n-1)$*-dimensional coordinate sector.*

Proof. (1°) Criterion (5) shows that the limit of a sequence of Chebyshev polynomials is a Chebyshev polynomial.

(3°) Since $T_n(\,\cdot\,, \pm 1) = (\pm 1)^n$, all Chebyshev polynomials lie in the $(n-1)$-dimensional plane

$$L_{n-1} := \left\{ P_n(x) = x^n + (x^2 - 1) \sum_{j=0}^{n-2} r_j x^j \right\}. \tag{20}$$

(4°) In the plane L_{n-1} we consider the open set Ω containing all Chebyshev polynomials and consisting of polynomials P_n of degree precisely n whose derivatives have simple zeros that all lie in the interval I:

$$-1 < x_1 < x_2 < \dots < x_{n-1} < 1, \qquad \dot{P}_n(x_s) = 0. \tag{21}$$

The map $\Phi\colon \Omega \to \mathbb{R}^{n-1}$ defined by the formula

$$\Phi(P_n) = \{y_s\}_{s=1}^{n-1}, \qquad y_s := (-1)^{n+s} P_n(x_s), \tag{22}$$

is a real-analytic diffeomorphism. For the map $\{r_k\}_{k=0}^{n-2} \to \{x_s\}_{s=1}^{n-1}$ is real analytic in the domain where the leading coefficient r_{n-2} is distinct from zero and all the x_s are distinct. Hence the map $\{r_k\}_{k=0}^{n-2} \to \{y_s\}_{s=1}^{n-1}$ is also real analytic. The map Φ has maximum rank since its Jacobian matrix

$$\frac{\partial y_s}{\partial r_k} = (-1)^{n+s}(x_s^2-1)x_s^k, \qquad s=1,\dots,n-1, \quad k=0,\dots,n-2,$$

is the product of a Vandermonde matrix and a non-singular diagonal matrix.

We claim that Φ is a bijection between the space of Chebyshev polynomials and the sector

$$\{y_s \geqslant 1,\ s=1,\dots,n-1\}. \tag{23}$$

The zeros x_s of the derivative of a Chebyshev polynomial T_n lie outside the intervals I_k, $k=1,\dots,n$, so that $(-1)^{n+s}T_n(x_s) \geqslant 1$. Conversely, let $\Phi(P_n) \in [1,\infty)^{n-1}$. The zeros of the derivative partition the interval I into n intervals of monotonicity of P_n (see (21)). For each of these n intervals the value of P_n is at least 1 at one end-point and at most -1 at the other, therefore $P_n = T_n$ by Theorem 1. The injectivity and the surjectivity of the restriction of Φ to $\bigcup_E T_n(E,\,\cdot\,)$ are consequences of the relations connecting this map and the above-introduced Akhiezer coordinates $\{\widetilde{\nu}_k\}_{k=1}^{n-1}$ in the space of Chebyshev polynomials:

$$y_s = \cosh \widetilde{\nu}_s, \qquad s=1,\dots,n-1.$$

The required homeomorphism $[1,\infty)^{n-1} \to \bigcup_E T_n(E,\,\cdot\,)$ is realized by the restriction of the map Φ^{-1} to the coordinate sector. We have already shown that this map is well defined, real analytic, and has maximum rank up to the boundary of the sector.

(2°) The norm $\|T_n(\,\cdot\,,x)\|_{C(\overline{I})} = \max_s y_s$ is unbounded in the space of Chebyshev polynomials because the map $\Phi\big|_{\bigcup_E T_n(E,\,\cdot\,)}$ is surjective.

2.1. Abel's equations.

Proof of Theorem 3. (1°) Let $T_n(E^+,x)$ be a Chebyshev polynomial with support set E^+. By Theorem 1 the holomorphic map $X{:}\ = \mathbb{CP}_1 \xrightarrow{T_n} \mathbb{CP}_1 := S$ has branching order 2 at all points in the inverse image of $\pm 1 \in S$ except for $2g+2$ points in ∂E^+, where it is unbranched. We consider now the surfaces $\widetilde{X}$ and $\widetilde{S}$ that two-sheetedly cover X and S, respectively. The surface $\widetilde{X} := M(E^+)$ has ramification points of order 2 over points in ∂E^+ and $\widetilde{S} := \mathbb{CP}_1$ has second-order ramification points over $\pm 1 \in S$:

$$\text{(a)}\quad \begin{array}{rcl} M(E^+) =: & \widetilde{X} & \xrightarrow{\widetilde{T}_n} \widetilde{S} := \mathbb{CP}_1 \\ & {\scriptstyle\chi}\big\downarrow & \big\downarrow{\scriptstyle\sigma} \\ \mathbb{CP}_1 =: & X & \xrightarrow{T_n} S := \mathbb{CP}_1 \end{array}, \qquad \text{(b)}\quad \begin{array}{c} \widetilde{X}_\circ \xrightarrow{\widetilde{T}_n} \widetilde{S}_\circ \\ {\scriptstyle\chi}\big\downarrow \qquad \big\downarrow{\scriptstyle\sigma} \\ X_\circ \xrightarrow{T_n} S_\circ \end{array}. \tag{24}$$

The hyperelliptic involution $J{:}\ (x,w) \to (x,-w)$, $(x,w) \in M$ generates the covering group $\mathfrak{G}(\chi)$ of the branched cover $\chi{:}\ (x,w) \to x$. The covering group $\mathfrak{G}(\sigma)$ of

the second branched cover $\sigma\colon \tilde{s} \to \frac{1}{2}(\tilde{s} + 1/\tilde{s})$ is generated by the map $\tilde{s} \to 1/\tilde{s}$, $\tilde{s} \in \widetilde{S}$. We claim that T_n can be lifted to the covering spaces, that is, there exists a map $\widetilde{T}_n\colon \widetilde{X} \to \widetilde{S}$ such that the diagram (24a) is commutative.

We define first a narrowed-down map $\widetilde{T}_n\colon \widetilde{X}_\circ \to \widetilde{S}_\circ$ such that σ is unbranched on its range $\widetilde{S}_\circ \subset \widetilde{S}$. We consider the spaces

$$\widetilde{S}_\circ = S_\circ := \mathbb{CP}_1 \setminus \{\pm 1\}, \quad X_\circ := \mathbb{CP}_1 \setminus \{T_n^{-1}(\pm 1)\}, \quad \widetilde{X}_\circ := \chi^{-1}(X_\circ),$$

in which we select distinguished points in a coordinated manner, taking account of the maps χ, T_n, σ in (24b). We define the map $\widetilde{T}_n\colon \widetilde{X}_\circ \to \widetilde{S}_\circ$ by associating the end-point of each path $\tau \subset \widetilde{X}_\circ$, starting at the distinguished point, with the end-point of the lifted path $\sigma^{-1} \circ T_n \circ \chi(\tau) \subset \widetilde{S}_\circ$ starting at the corresponding distinguished point. This map is well defined if we have the embedding of fundamental groups [17]

$$(T_n \circ \chi)\pi_1(\widetilde{X}_\circ) \subset \sigma\pi_1(\widetilde{S}_\circ). \tag{25}$$

We now verify this embedding. The fundamental group $\pi_1(S_\circ)$ is a free one-generator group. Its subgroup $\sigma\pi_1(\widetilde{S}_\circ)$ is generated by the *square* of the generator. We can choose in $\pi_1(\widetilde{X}_\circ)$ a basis with elements representable by two types of loop: the $4n$ loops of the first type enclose the punctures $\chi^{-1} \circ T_n^{-1}(\pm 1)$ in the Riemann surface $M(E^+)$, while the $2g$ loops of the second type are homotopic to the cycles $\mathbf{A}_1, \ldots, \mathbf{A}_g; \widetilde{\mathbf{B}}_1, \ldots, \widetilde{\mathbf{B}}_g$ (see Fig. 1). The branching order of $T_n \circ \chi$ at each puncture of $\widetilde{X}_\circ$ is 2, therefore the projections of the loops of the first type make *two* circuits about punctures of $S_\circ$. The projection onto $X_\circ$ of a loop of the second kind encloses (see Fig. 1) an *even* number of points in ∂E^+ (11) as well as some other points in the inverse image of $T_n^{-1}(\pm 1)$. We can deform this projection in $X_\circ$ into a product of simple loops enclosing punctures of $X_\circ$. The projection of this product onto $S_\circ$ is homotopic to an *even* power of the generator of $\pi_1(S_\circ)$.

Thus, we have defined $\widetilde{T}_n$ on the entire surface $\widetilde{X}$ except for $4n$ punctures at which we can define it by continuity. Analyzing the dependence of the above lifting construction on the distinguished points in the spaces in the diagram (24b) we conclude that the solution $\widetilde{T}_n$ is unique up to translations in $\mathfrak{G}(\sigma)$ and satisfies the equivariance relation

$$\widetilde{T}_n(Jt) = \frac{1}{\overline{\widetilde{T}_n(t)}}, \qquad t \in \widetilde{X}. \tag{26}$$

On the cover $\widetilde{X} := M(E^+)$ we consider the Abelian differential of the third kind

$$d\eta = \frac{d\widetilde{T}_n}{n\widetilde{T}_n}; \tag{27}$$

in view of the commutativity of (24a), it has the same poles as $T_n \circ \chi$, that is, it has poles at the points ∞_+ and ∞_- at infinity in the upper and the lower sheets of $M(E^+)$. We specialize $\widetilde{T}_n$ by setting

$$\operatorname{Res} d\eta(t)\big|_{t=\infty_\pm} = \mp 1. \tag{28}$$

It remains to find the periods of $d\eta$. Deforming the contour $\mathbf{A}_s$ into the interval $[a_s, b_s]$ passed first in the lower sheet of M, and then — in the opposite direction — in the upper sheet, we see that

$$\int_{\mathbf{A}_s} d\eta = \frac{1}{n} \ln \widetilde{T}_n\big|_{\mathbf{A}_s} = \frac{i}{n} \arg \widetilde{T}_n\big|_{\mathbf{A}_s} = 0, \qquad s = 1, \dots, g.$$

In a similar way we deform $\widetilde{\mathbf{B}}_k$ into the interval $[b_k, a_{k+1}]$ passed along the lower bank first and along the upper bank after that. As $\widetilde{x} \in \widetilde{X}$ moves along this contour, the point $\tilde{s} := \widetilde{T}_n(\widetilde{x})$ makes m_k counterclockwise circuits on the circle $|\tilde{s}| = 1$, therefore

$$\int_{\widetilde{\mathbf{B}}_k} d\eta = \frac{i}{n} \arg \widetilde{T}_n\big|_{\widetilde{\mathbf{B}}_k} = 2\pi i \, \frac{m_k}{n}, \qquad k = 0, 1, \dots, g.$$

The diagram (24a) is commutative, which gives us the representation (16): if $x = \chi(t)$, then

$$T_n(E^+, x) = T_n(E^+, \chi(t)) = \sigma \circ \widetilde{T}_n(t) = \sigma \circ \exp\biggl(n \int_{t_0}^{t} d\eta\biggr) = \cos\biggl(in \int_{(1,0)}^{(x,w)} d\eta\biggr).$$

(2°) Conversely, let $M(E^+)$ be a Riemann surface of the form (12) such that relations (15) hold for an $\mathbf{A}$-normalized differential $d\eta$ of the third kind. In view of (15), we have on $M(E^+)$ the well-defined function

$$\widetilde{T}_n(t) = \exp\biggl(n \int_{t_0}^{t} d\eta\biggr), \qquad t \in M, \quad t_0 = (1, 0) \in M,$$

satisfying equivariance condition (26) because $d\eta$ changes sign after the involution J, which fixes t_0. The function $\widetilde{T}_n$ gives rise to the map of quotient spaces

$$T_n \colon X = \widetilde{X}/\mathfrak{G}(\chi) \xrightarrow{\widetilde{T}_n} \widetilde{S}/\mathfrak{G}(\sigma) = S,$$

which can be parametrically expressed as follows:

$$T_n(x) = \sigma \circ \widetilde{T}_n(t) = \cos\biggl(ni \int_{t_0}^{t} d\eta\biggr), \qquad x = \chi(t) \in X, \quad t \in \widetilde{X},$$

and which is a rational function with a single pole of order n at infinity, that is, a polynomial.

It remains to show that the interval (b_k, a_{k+1}) consists of m_k subintervals mapped onto I by the function $T_n(x)$ in a one-to-one manner. In fact, as x decreases from a_{k+1} to b_k, the argument of the cosine in (16) changes monotonically (because the differential (14) is of constant sign on this interval) by πm_k. At the end-points b_k and a_{k+1} the values of this argument are multiples of π; this can be deduced by induction on k from the $\mathbf{A}$-normalization of $d\eta$. Since the sum of the m_k is equal to the degree of $T_n(x)$, it follows by Theorem 1 that T_n is a Chebyshev polynomial with support set E^+.

2.2. Geometry of the manifold of support sets. We have already associated with each Chebyshev polynomial T_n its support set E^+ and the hyperelliptic curve $M(E^+)$. We shall now study differential-geometric properties of the map of the cell $\mathbb{T}(m_0, \dots, m_g)$ into the space of two-sheeted surfaces of genus g.

Definition. By the *moduli space* $\mathcal{H}(\mathbb{R}; g, 1)$ associated with the problem of least deviation we mean the $2g$-dimensional open simplex (11) in the space of the variables a_s, b_s the points in which are identified with Riemann surfaces M with two homology bases described above.[1] The solutions of Abel's equations (15) with n as in (9) make up the *set of support sets*, denoted also by $\mathbb{T}(m_0, m_1, \dots, m_g) \subset \mathcal{H}(\mathbb{R}; g, 1)$: it will always be clear from the context whether we speak about a set of polynomials or about a subset of the moduli space.

Theorem 5 [12]**.** *For each collection of positive integers $(m_0, m_1, \dots, m_g)$ the set $\mathbb{T}(m_0, m_1, \dots, m_g)$ is a g-dimensional non-singular real analytic submanifold of $\mathcal{H}(\mathbb{R}; g, 1)$ homeomorphic to a cell. Distinct submanifolds $\mathbb{T}$ are either disjoint or the same; in the latter case the corresponding collections $(m_0, \dots, m_g)$ are proportional. The union of $\mathbb{T}$-submanifolds is dense in the moduli space.*

We now formulate a lemma of which Theorem 5 is a consequence. Let $d\eta$ be an Abelian differential (14) of the third kind (not necessarily normalized) on a Riemann surface (12). We define a map $\{a_k, b_k, c_k\}_{k=1}^g \to \{\lambda_k, \mu_k, \nu_k\}_{k=1}^g$ by the equalities

$$\lambda_k := \int_{\mathbf{A}_k} d\eta, \quad \mu_k := -i \int_{\mathbf{B}_k} d\eta, \quad \nu_k := \int_{\mathbf{C}_k} d\eta, \qquad k = 1, \dots, g; \tag{29}$$

the paths $\mathbf{A}_k, \mathbf{C}_k$ are as in Fig. 1, and $\mathbf{B}_k := \widetilde{\mathbf{B}}_0 + \widetilde{\mathbf{B}}_1 + \dots + \widetilde{\mathbf{B}}_{k-1}$.

Lemma 1. *The map* (29) *is a real-analytic diffeomorphism of the $3g$-dimensional simplex*

$$-1 < a_1 < c_1 < b_1 < a_2 < \dots < c_g < b_g < 1 \tag{30}$$

onto the product of the g-dimensional simplex and the $2g$-dimensional cone described by the inequalities

$$\begin{gathered} 0 < \mu_1 < \mu_2 < \dots < \mu_g < 2\pi, \\ \lambda_k < \nu_k, \quad 0 < \nu_k, \qquad k = 1, \dots, g. \end{gathered} \tag{31}$$

Proof. The Schwarz–Christoffel integral

$$2i \int_{-1}^{x} d\eta, \qquad \operatorname{Im} x > 0,$$

maps the upper half-plane conformally onto a 'comb-step' domain similar to the one depicted in Fig. 2. The parameters of this domain are in one-to-one correspondence with $\{\lambda_k, \mu_k, \nu_k\}$; for instance, the x-coordinate of the kth slit is μ_k, the height of the kth slit over the horizontal platform following next is ν_k, and the overfall of the horizontal platforms adjoining the kth slit is λ_k. We can define the inverse map from the domain (31) onto the simplex (30) by means of a suitably normalized conformal map of the 'comb-step' domain in Fig. 2 onto the upper half-plane. The injectivity

[1] The space $\mathcal{H}$ also contains conformally equivalent surfaces.

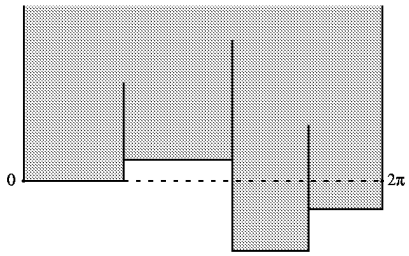


Figure 2. A 'comb-step' domain for $g = 3$

of the real-analytic map $\{a_k, b_k, c_k\} \to \{\lambda_k, \mu_k, \nu_k\}$ is a simple consequence of the uniqueness of the normalized conformal map.

We claim that the bijective map $\{a_k, b_k, c_k\} \longleftrightarrow \{\lambda_k, \mu_k, \nu_k\}$ has maximum rank. Differentiating the expressions for λ, μ, ν with respect to a, b, c we obtain the following expression for the Jacobian matrix of (29):

$$\frac{\partial(\lambda,\mu,\nu)}{\partial(a,b,c)} = \left\| \begin{array}{cccccc} \frac{1}{2}\int_{\mathbf{A}_k} \frac{d\eta}{x-a_s} & \dots & \frac{1}{2}\int_{\mathbf{A}_k} \frac{d\eta}{x-b_s} & \dots & -\int_{\mathbf{A}_k} \frac{d\eta}{x-c_s} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{2i}\int_{\mathbf{B}_k} \frac{d\eta}{x-a_s} & \dots & \frac{1}{2i}\int_{\mathbf{B}_k} \frac{d\eta}{x-b_s} & \dots & -\frac{1}{i}\int_{\mathbf{B}_k} \frac{d\eta}{x-c_s} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{2}\int_{\mathbf{C}_k} \frac{d\eta}{x-a_s} & \dots & \frac{1}{2}\int_{\mathbf{C}_k} \frac{d\eta}{x-b_s} & \dots & -\int_{\mathbf{C}_k} \frac{d\eta}{x-c_s} \end{array} \right\|,$$

$k, s = 1, \dots, g$. If this matrix is singular, then there exists a differential of the second kind

$$d\omega := \sum_{s=1}^{g} \left(\frac{\alpha_s}{x-a_s} + \frac{\beta_s}{x-b_s} + \frac{\gamma_s}{x-c_s} \right) d\eta$$

with constant coefficients α_s, β_s, and γ_s such that its integrals over all the curves $\mathbf{A}_k$, $\mathbf{B}_k$, and $\mathbf{C}_k$ vanish. Note that $d\omega$ can have poles only at the points $(a_s, 0)$, $(b_s, 0) \in M$, and the residues of $d\omega$ at the poles must be equal to zero because $d\omega$ changes sign after the involution J, which fixes the ramification points of M. The Abelian integral

$$\omega(t) = \int_{t_0}^{t} d\omega, \qquad t_0 = (1, 0) \in M, \tag{32}$$

is a single-valued function on M because the periods of $d\omega$ over the basis cycles and over cycles about the poles are equal to zero. The function ω changes sign after the involution J because the differential $d\omega$ is odd, therefore $\omega = 0$ at both points $t_k \in M$ lying over c_k, $k = 1, \dots, g$. The function $w\omega \in \mathbb{C}(M)$ is J-invariant, and its only singularities are the two poles at infinity of order at most $g + 1$, therefore

$$w(t)\omega(t) = P_{g+1}(x), \qquad x = \chi(t), \quad t \in M,$$

where P_{g+1} is a polynomial of degree $g + 1$, which, however, has the $g + 2$ zeros $\pm 1, c_1, \dots, c_g$. Hence $\omega \equiv 0$ and the Jacobian matrix is non-singular.

Proof of Theorem 5. We embed the moduli space $\mathcal{H}(\mathbb{R}; g, 1)$ in the $3g$-dimensional simplex (30) by complementing the system $\{a_k, b_k\}_{k=1}^g$ by the zeros c_k of the **A**-normalized differential of the third kind $d\eta$. The correspondence between $\mathcal{H}$ and the manifold $\{\lambda_k(a, b, c) = 0\}_{k=1}^g$ is one-to-one. We claim that the restriction of the projection $\{a_k, b_k, c_k\}_{k=1}^g \to \{a_k, b_k\}_{k=1}^g$ to the zero set of $\lambda_k(a, b, c)$ has maximum rank.

The cotangent space to $\{\lambda_k(a, b, c) = 0\}_{k=1}^g$ is spanned by the differentials da_k, db_k, dc_k, with the constraint that

$$\sum_{j=1}^{g}\biggl(\int_{\mathbf{A}_k}\frac{d\eta}{x-a_j}\biggr)da_j+\sum_{j=1}^{g}\biggl(\int_{\mathbf{A}_k}\frac{d\eta}{x-b_j}\biggr)db_j = 2\sum_{j=1}^{g}\biggl(\int_{\mathbf{A}_k}\frac{d\eta}{x-c_j}\biggr)dc_j, \qquad k=1,\dots,g. \tag{33}$$

The matrix $\left\|\int_{\mathbf{A}_k}(x-c_j)^{-1}\,d\eta\right\|$ is non-degenerate, for otherwise there exists an Abelian differential of the first kind $\sum_{j=1}^g \gamma_j(x-c_j)^{-1}\,d\eta$ with **A**-periods zero. Hence the map of the cotangent spaces induced by the projection is non-degenerate.

The subset $\mathbb{T}(m_0, \dots, m_g)$ of $\mathcal{H}(\mathbb{R}; g, 1)$ is described in (30) by g additional equalities $\mu_k = 2\pi\widetilde{m}_k/n$, $k = 1, \dots, g$, where n is as in (9). Hence all the assertions of the theorem follow easily by Lemma 1.

2.3. Deformations of Chebyshev polynomials. We now study the question of the stability of the representation (16) under small variations of the variables $\{a, b\}$ in the moduli space $\mathcal{H}(\mathbb{R}; g, 1)$. The main result here is a consequence of Theorem 6: *the function*

$$T_n^*(E^+, x) := \cos\biggl(in\int_1^x d\eta(E^+)\biggr) \tag{34}$$

represents a Chebyshev polynomial with the same accuracy with which E^+ *satisfies Abel conditions* (15). The same conclusion follows from numerical calculations. Calculating with accuracy 10^{-13} one can achieve that (15) holds with accuracy 10^{-11}–10^{-12}, in which case the function T_n^* will be equal to its Lagrange interpolation polynomial with accuracy 10^{-10}–10^{-11} for various collections of points of interpolation.

For an arbitrary point $\{a_k, b_k\}_{k=1}^g$ in the moduli space we fix the **A**-normalized Abelian differential $d\eta(E^+, x)$ (14), which is single-valued in the plane with slits $(-\infty, -1] \cup \bigl\{\bigcup_{k=1}^g [a_k, b_k]\bigr\} \cup [1, +\infty)$ and the sign of which we choose from the condition $\int_{-1}^{1} d\eta(E^+, x) = -i\pi$. The corresponding function $T_n^*(E^+, x)$ is not an x-polynomial in general. The function T_n^* takes equal values on the opposite sides of the cuts $(-\infty, -1]$ and $[1, +\infty)$, therefore it is x-holomorphic in $\mathbb{C} \setminus \bigcup_{k=1}^g [a_k, b_k]$ and has a pole of order n at infinity.

We now study the dependence of the function (34) on the parameter E^+. The function T_n^* is real analytic in a, b and holomorphic in x in the $(2g+2)$-dimensional

domain

$$\left\{\{a,b,x\} : \{a_k,b_k\}_{k=1}^g \in \mathcal{H}(\mathbb{R};g,1),\ x\in\mathbb{C}\setminus\bigcup_{k=1}^g [a_k,b_k]\right\}. \tag{35}$$

Its differential is as follows:

$$\begin{aligned} dT_n^*(E^+,x) = -in\sin\left(ni\int_1^x d\eta(E^+,u)\right) \\ \times\left[\dot\eta(E^+,x)\,dx + \sum_{j=1}^g \frac{1}{2}\left(\int_1^x \frac{d\eta(E^+,u)}{u-a_j}\right)da_j\right. \\ \left. + \sum_{j=1}^g \frac{1}{2}\left(\int_1^x \frac{d\eta(E^+,u)}{u-b_j}\right)db_j - \sum_{j=1}^g\left(\int_1^x \frac{d\eta(E^+,u)}{u-c_j}\right)dc_j\right]. \end{aligned} \tag{36}$$

The integration here proceeds on the surface $\{a,b\}=\text{const}$, and the coefficients of the differential form are independent of the integration path thanks to conditions (33), which hold in view of the **A**-normalization of $d\eta$. One can verify that the singularities of the coefficients of the form dT_n^* for $x=\pm1$ are merely 'imaginary'.

Theorem 6. *The function $T_n^*(E^+,x)$ is differentiable with respect to $\{a,b,x\}$ at each point $\{a^0,b^0,x^0\}$ such that $\{a^0,b^0\}\in\mathbb{T}(m_0,m_1,\dots,m_g)$ and x^0 lies in the closure of the domain $\mathbb{C}\setminus\bigcup_{k=1}^g[a_k,b_k]$ such that a sphere with slits is completed by the two banks of each slit.*[2] *For $x^0\notin\{a_k^0,b_k^0\}_{k=1}^g$ the differential dT_n^* is well defined and coincides with* (36)*, while if $x^0\in\{a_k^0,b_k^0\}_{k=1}^g$, then*

$$dT_n^*(E_0^+,x^0) = (-1)^{n+\widetilde{m}_k}2n^2\operatorname*{Res}_{u=x^0}\dot\eta^2(E_0^+,u)\cdot\begin{cases} dx-da_k & \text{for } x^0=a_k^0,\\ dx-db_k & \text{for } x^0=b_k^0.\end{cases} \tag{37}$$

Proof. At an arbitrary point $\{a_k,b_k\}_{k=1}^g$ in the moduli space the function $T_n^*(E^+,x)$ can be continued analytically with respect to x across the slits $\bigcup_{k=1}^g[a_k,b_k]$. Accordingly, the differential dT_n^* is well defined at both banks of the slits and is as in (36). This differential has singularities at the ramification points of T_n^*, which vanish on the $\mathbb{T}$-manifolds.

For instance, assume that $x^0=a_k^0$. We transform the expression for the Abel integral in (34) as follows:

$$\begin{aligned} \int_1^x d\eta(E^+) &= \int_1^{a_k} d\eta(E^+) + \int_{a_k}^x d\eta(E^+) \\ &= i\pi - i\,\frac{\mu_k(E^+)}{2} \\ &\quad \pm\int_{a_k}^x (u-a_k)^{-1/2}\left(\sqrt{\operatorname*{Res}_{u=a_k^0}\dot\eta^2(E_0^+,u)} + O(|\delta a|+|\delta b|+|u-a_k|)\right)du \\ &= i\pi\frac{m_k+\dots+m_g}{n} \\ &\quad \pm 2\sqrt{x-a_k}\sqrt{\operatorname*{Res}_{u=a_k^0}\dot\eta^2(E_0^+,u)} + O(|\delta a|+|\delta b|+|\delta x|), \end{aligned}$$

[2] We mean here the closure with respect to the distance measured within the domain (that is, in the Mazurkiewicz metric).

where $|\delta a| := \sum_{j=1}^{g} |a_j - a_j^0|$, $|\delta b| := \sum_{j=1}^{g} |b_j - b_j^0|$, $|\delta x| := x - x^0$, and the signs of the root functions are chosen depending on the bank of the slit containing x. The variation of T_n^* is as follows:

$$\begin{aligned}\delta T_n^* &:= T_n^*(E^+, x) - T_n^*(E_0^+, x^0)\\ &= (-1)^{m_k + \cdots + m_g}\\ &\quad \times \Big[\cos\Big(\pm 2ni\sqrt{x - a_k}\sqrt{\operatorname*{Res}_{u=a_k^0} \dot\eta^2(E_0^+, u)} + O(|\delta a| + |\delta b| + |\delta x|)\Big) - 1\Big]\\ &= (-1)^{n+\widetilde{m}_k} 2n^2 \operatorname*{Res}_{u=a_k^0} \dot\eta^2(E_0^+, u)(\delta x - \delta a_k) + o(|\delta a| + |\delta b| + |\delta x|).\end{aligned}$$

§ 3. Schottky model

We showed in § 2 that the problem of the effective calculation of Chebyshev polynomials $T_n(\,\cdot\,,\cdot\,)$ can be reduced to finding representations (models) of Riemann surfaces M making up the moduli space $\mathcal{H}(\mathbb{R}; g, 1)$ such that

(A) the constraint (15) is effectively resolved,
(B) the expression (16) for T_n and the dependence of the variable x on a point in M are effectively calculated.

We shall take for such a representation the Schottky uniformization. We supply all necessary information about that model in the present section.

3.1. Symmetric Schottky group. Let $\mathbf{C}_1, \dots, \mathbf{C}_g$ be circles with centres on the real axis that lie in the right half-plane outside one another (see Fig. 3).

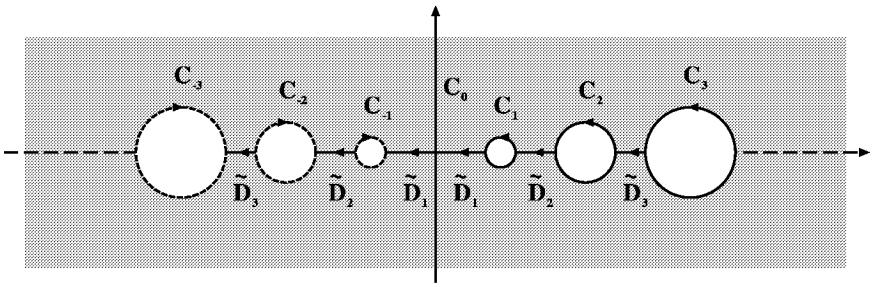


Figure 3. A circular domain $\mathcal{R}$ with boundary $\sum_{s=1}^{g}(\mathbf{C}_{-s} + \mathbf{C}_s)$, $g = 3$

The set of motions G of the Riemann sphere that are products of even numbers of reflections relative to the imaginary axis $\mathbf{C}_0$ and the circles $\mathbf{C}_1, \dots, \mathbf{C}_g$ is the Schottky group $\mathfrak{G}$; its fundamental domain $\mathcal{R}(\mathfrak{G})$ is the exterior of the circles $\mathbf{C}_1, \dots, \mathbf{C}_g$ and the circles $\mathbf{C}_{-1}, \dots, \mathbf{C}_{-g}$ symmetric to them relative to $\mathbf{C}_0$ (see [18]). The components of the boundary of the fundamental domain are identified with one another by means of the linear fractional maps

$$G_k u := G_k(u) := c_k - \frac{r_k^2}{u + c_k}, \qquad k = 1, \dots, g, \tag{38}$$

which are free generators of $\mathfrak{G}$; here c_k is the centre and r_k the radius of the circle $\mathbf{C}_k$. We introduce the norm $|G|$ in the Schottky group in the standard way as the length of the non-cancellable representation of an element G in terms of the generators $G_1^{\pm1}, G_2^{\pm1}, \dots, G_g^{\pm1}$.

On the Riemann surface $\mathcal{D}(\mathfrak{G})/\mathfrak{G}$ (here $\mathcal{D}$ is the region of discontinuity of $\mathfrak{G}$ [18]) we fix the canonical basis $\mathbf{C}_1, \dots, \mathbf{C}_g$; $\mathbf{D}_1, \dots, \mathbf{D}_g$ in the homology group, where $\mathbf{D}_j := \widetilde{\mathbf{D}}_1 + \dots + \widetilde{\mathbf{D}}_j$ and we indicate representatives of $\mathbf{C}_j$ and $\widetilde{\mathbf{D}}_j$ in Fig. 3.

3.2. Functions and differentials in the region of discontinuity. It turns out that the Poincaré Θ_1-series for the above-defined symmetric Schottky group $\mathfrak{G}$ converge absolutely [19] and uniformly on compact subsets of the region of discontinuity $\mathcal{D}(\mathfrak{G})$. This enables one to construct an 'effective' function theory on the Riemann surface $\mathcal{D}(\mathfrak{G})/\mathfrak{G}$. We start from an Abelian differential of the third kind

$$\begin{aligned} d\eta_{xy}(u) :=& \sum_{G\in\mathfrak{G}} \{(Gu-x)^{-1} - (Gu-y)^{-1}\}\, dG(u) \\ \overset{(42)}{=}& \sum_{G\in\mathfrak{G}} \{(u-Gx)^{-1} - (u-Gy)^{-1}\}\, du, \end{aligned} \tag{39}$$

where $x, y \in \mathcal{D}(\mathfrak{G})$. Integrating the series (39) termwise over the circles $\mathbf{C}_1, \dots, \mathbf{C}_g$ we can verify that if points x and y lie in the above-described fundamental domain $\mathcal{R}(\mathfrak{G})$, then $d\eta_{xy}$ is a $\mathbf{C}$-normalized differential of the third kind on the surface $M = \mathcal{D}(\mathfrak{G})/\mathfrak{G}$, that is,

$$\int_{\mathbf{C}_s} d\eta_{xy} = 0, \qquad x, y \in \mathcal{R}, \quad s = 1, \dots, g. \tag{40}$$

The easiest way to obtain a differential of the first kind is to put the poles x and y in (39) in the same orbit of $\mathfrak{G}$:

$$\begin{aligned} d\zeta_k(u) := d\eta_{G_k y y} &= \sum_{G\in\mathfrak{G}} \left\{(u-GG_ky)^{-1} - (u-Gy)^{-1}\right\} du \\ &= \sum_{G\in\mathfrak{G}|\langle G_k\rangle} \sum_{m=-\infty}^{\infty} \left\{(u-GG_k^{m+1}y)^{-1} - (u-GG_k^{m}y)^{-1}\right\} du \\ &= \sum_{G\in\mathfrak{G}|\langle G_k\rangle} \left\{(u-G\alpha_k)^{-1} - (u-G\beta_k)^{-1}\right\} du \\ &= \sum_{G\in\langle G_k\rangle|\mathfrak{G}} \left\{(Gu-\alpha_k)^{-1} - (Gu-\beta_k)^{-1}\right\} dG(u), \qquad k = 1, \dots, g, \end{aligned} \tag{41}$$

here α_k and β_k are the attracting and repelling fixed points of G_k and we take the sum over some representatives of the cosets of $\mathfrak{G}$ by its subgroup $\langle G_k\rangle$ generated by the element $G_k \in \mathfrak{G}$. The terms in (41) are independent of our choice of these representatives; this follows from the infinitesimal form of the cross-ratio identity

$$\frac{\dfrac{d}{du}G(u)(\alpha-\beta)}{(Gu-\alpha)(Gu-\beta)} = \frac{G^{-1}\alpha - G^{-1}\beta}{(u-G^{-1}\alpha)(u-G^{-1}\beta)}\,. \tag{42}$$

We observe (by integrating the series termwise) that the $d\zeta_k$, $k = 1, \dots, g$, make up a $\mathbf{C}$-normalized collection of differentials, that is,

$$\int_{\mathbf{C}_s} d\zeta_k = 2\pi i \delta_{sk}, \qquad s, k = 1, \dots, g. \tag{43}$$

The Abelian integrals η_{xy} and ζ_k are many-valued in the region of discontinuity $\mathcal{D}(\mathfrak{G})$, but their exponentials are single-valued:

$$\exp \int_v^u d\eta_{xy} = \prod_{G \in \mathfrak{G}} \frac{u - Gx}{u - Gy} : \frac{v - Gx}{v - Gy} =: (u, v; x, y), \tag{44}$$

$$\exp \int_\infty^u d\zeta_k = \prod_{G \in \mathfrak{G} | \langle G_k \rangle} \frac{u - G\alpha_k}{u - G\beta_k} =: E_k(u), \qquad k = 1, \dots, g. \tag{45}$$

The functions $(u, v; x, y)$ and $E_k(u)$ were introduced by Schottky [20] and, as functions of u, are 'Prym functions' on the Riemann surface $M = \mathcal{D}(\mathfrak{G})/\mathfrak{G}$: going around a cycle in M multiplies them by constants:

$$(G_k u, v; x, y) = (u, v; x, y) \frac{E_k(x)}{E_k(y)}, \tag{46}$$

$$E_k(G_j u) = E_k(u) E_{kj}. \tag{47}$$

The constant E_{kj}, the exponential of the period of an Abelian integral of the first kind, has the representation

$$E_{kj} = E_{jk} = \prod_{G \in \langle G_k \rangle | \mathfrak{G} | \langle G_j \rangle} \frac{G\alpha_j - \alpha_k}{G\alpha_j - \beta_k} : \frac{G\beta_j - \alpha_k}{G\beta_j - \beta_k}, \qquad k, j = 1, \dots, g, \tag{48}$$

where the product is taken for bilateral cosets of the group $\mathfrak{G}$ and if $j = k$, then the coefficient $0/\infty$ corresponding to $G = 1$ must be replaced by the dilation coefficient $\lambda_k := \dot{G}_k(\alpha_k)$.

Each single-valued meromorphic function $F(u)$ on M (an automorphic function) has a representation

$$F(u) = F(v) \prod_{k=1}^{s} (u, v; x_k, y_k) \prod_{k=1}^{g} \left(\frac{E_k(u)}{E_k(v)} \right)^{l_k}, \tag{49}$$

where the x_k are the zeros and the y_k are the poles of $F(u)$, v is an arbitrary point in $\mathcal{D}(\mathfrak{G})$, and $l_k \in \mathbb{Z}$. In fact, expanding the Abelian differential dF/F into a sum of differentials of the third kind $d\eta_{x_k y_k}$ and differentials of the first kind $d\zeta_k$, integrating from v to u, and exponentiating we arrive at (49). The restrictions on the divisor F imposed by Abel's theorem are just the conditions that the right-hand side of (49) must remain the same after going around $\mathbf{D}$-cycles.

3.3. Calculation of functions and differentials. In this subsection we give an estimate of the remainder in the series (39) that is uniform on compact sets, demonstrates the convergence of all series and products in § 3.2, and enables one to find also estimates for their remainders.

Computational practice shows that Poincaré Θ_1-series converge slowly if the intervals between successive circles in the sequence $\mathbf{C}_{-g},\dots,\mathbf{C}_{-1},\mathbf{C}_1,\dots,\mathbf{C}_g$ are small compared with their radii. We shall introduce a quantity $q_{\max}$ characterizing the convergence rate of the Θ_1-series for a fixed group $\mathfrak{G}$.

Definition. Let $z_k^{\pm} := c_k \pm r_k$, $k = 1,\dots,g$, be the intersection points of the circles $\mathbf{C}_k$ and the real axis. We set

$$
\begin{aligned}
q_1 &:= \frac{z_2^- - z_1^-}{z_2^- - z_1^+} : \frac{z_1^-}{c_1}\,;\\
q_k &:= \frac{z_{k+1}^- - z_k^-}{z_{k+1}^- - z_k^+} : \frac{z_{k,1}^+ - z_k^-}{z_{k-1}^+ - z_k^+}\,, \quad k = 2,\dots,g-1;\\
q_g &:= \frac{c_g}{z_g^+} : \frac{z_{g-1}^+ - z_g^-}{z_{g-1}^+ - z_g^+}\ :
\end{aligned}
\tag{50}
$$

$$
1 < q_{\max} := \max_{k=1,\dots,g} q_k. \tag{51}
$$

Lemma 2. *The remainder in the Poincaré Θ_1-series satisfies the asymptotic estimate*

$$
\begin{aligned}
&\sum_{|G|>k} \bigl|(Gu-x)^{-1} - (Gu-y)^{-1}\bigr||\dot G(u)|\\
&\qquad \leqslant 2\Biggl(\sum_{s=1}^{g} r_s\Biggr)\bigl[\operatorname{dist}^{-2}\{\mathcal{K},\Lambda(\mathfrak{G})\} + o(1)\bigr]\left(\frac{\sqrt{q_{\max}}-1}{\sqrt{q_{\max}}+1}\right)^{k}\bigl[\sqrt{q_{\max}}+1\bigr]
\end{aligned}
\tag{52}
$$

uniformly on compact subsets $\mathcal{K}$ of $\mathcal{D}(\mathfrak{G})$, where $\Lambda(\mathfrak{G}) := \mathbb{CP}_1 \setminus \mathcal{D}(\mathfrak{G})$ is the limit set of the group, $x, y \in \overline{\mathcal{R}}$, $u \in \mathcal{K}$, and $o(1) \to 0$ as $k \to \infty$.

Proof. We have already noted that cross-ratio identity (42) enables one to write the Poincaré Θ_1-series (39) in the following form:

$$
\Theta_1(u) = \sum_{G\in\mathfrak{G}} \bigl\{(u-Gx)^{-1} - (u-Gy)^{-1}\bigr\}. \tag{53}
$$

The compact set $\mathcal{K}$ is disjoint from the compact set $G\overline{\mathcal{R}}$ for elements of G of sufficiently large norm, therefore the corresponding terms of (53) are holomorphic on $\mathcal{K}$. The absolute value of the general term of (53) has the estimate

$$
\operatorname{dist}^{-2}(\mathcal{K}, G\overline{\mathcal{R}})|Gx - Gy|,
$$

and since $\operatorname{dist}(\mathcal{K}, G\overline{\mathcal{R}}) \geqslant \operatorname{dist}(\mathcal{K},\Lambda(\mathfrak{G})) - o(1)$, it follows that to prove the lemma we must find an estimate for the sum $\sum_{|G|>k}|Gx - Gy|$.

First, we prove that if $|G| \neq 0$, then the diameter of the outer circle of $G\mathcal{R}$ is at least $\dfrac{\sqrt{q_{\max}}+1}{\sqrt{q_{\max}}-1}$ times as large as the sum of the diameters of the $2g-1$ inner circles. Let $[w_0^-, w_0^+]$ be the smallest interval of the real axis containing the diameters of all the inner circles of $G\mathcal{R}$ and let $[w_1^+, w_1^-]$ be the diameter of the outer circle lying on the real axis. It is known that a geodesic segment of fixed *non-Euclidean* length in the Lobachevskiĭ–Poincaré plane has the largest *Euclidean* length in the case when it is concentric with the absolute (see Fig. 4).

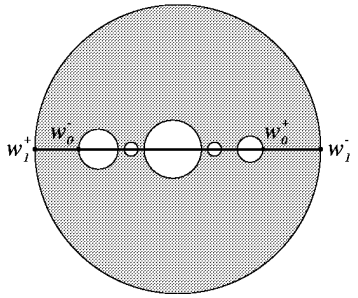


Figure 4. The outer boundary $G\mathcal{R}$ is the absolute of the Lobachevskiĭ–Poincaré plane

Applying this observation to $[w_0^-, w_0^+]$ we obtain

$$\frac{w_0^+ - w_0^-}{w_1^- - w_1^+} \leqslant \frac{\sqrt{\mu}-1}{\sqrt{\mu}+1}, \qquad \mu := \frac{w_1^- - w_0^-}{w_1^- - w_0^+} : \frac{w_1^+ - w_0^-}{w_1^+ - w_0^+};$$

in fact, μ is one of the cross-ratios listed in (50).

Denoting by $d(G\mathcal{R})$ the diameter of the outer circle $G\mathcal{R}$ we obtain the required estimate

$$\begin{aligned}\sum_{|G|>k} |Gx - Gy| &\leqslant \sum_{|G|>k} d(G\mathcal{R}) = \sum_{l=k+1}^{\infty} \sum_{|G|=l} d(G\mathcal{R}) \\ &\leqslant \sum_{|G|=k+1} d(G\mathcal{R}) \sum_{l=0}^{\infty} \left(\frac{\sqrt{q_{\max}}-1}{\sqrt{q_{\max}}+1}\right)^l \\ &\leqslant 2(\sqrt{q_{\max}}+1)\left[\frac{\sqrt{q_{\max}}-1}{\sqrt{q_{\max}}+1}\right]^k \left(\sum_{l=1}^{g} r_l\right).\end{aligned}$$

Corollary. *The series* $\dfrac{d}{du}\eta_{xy}$, $\dfrac{d}{du}\zeta_k$ *and the products* $(u, v; x, y)$, $E_k(u)$, *and* E_{kj} (see § 3.2) *are absolutely convergent.*

Proof. The series (41) for $\dot{\zeta}_k(u)$ is a (transformed) special case of the series $\dot{\eta}_{xy}$; the infinite products $E_k(u)$ and E_{kj} are special cases of the product $(u, v; x, y)$, which,

in turn, is obtained by exponentiating the integral of $d\eta_{xy}$. Hence it suffices to show that the expression (39) for $d\eta_{xy}$ converges uniformly on compact subsets of the region of discontinuity for $x, y \in \mathcal{D}(\mathfrak{G})$. The representation (53) shows that the series does not change if we apply a transformation $G \in \mathfrak{G}$ to its poles x and y, therefore each series (39) can be represented as a *finite* sum of series of the same form with poles $x, y \in \overline{\mathcal{R}}$. As concerns the latter, their uniform convergence on compact subsets is a consequence of the lemma.

Finally, we discuss briefly the general scheme of computation of sums and products taken over the group $\mathfrak{G}$. We construct the Caley graph of the Schottky group by putting the elements of the group at the vertices of a tree and by joining an element G with elements of the form $G_s^{\pm 1}G$, $s = 1, \dots, g$, one of which (provided that $|G| \neq 0$) is an element from the upper level (of norm $|G| - 1$) and $2g - 1$ are elements from the lower level (of norm $|G| + 1$). To calculate some term in the series (53) at a vertex G_sG it suffices to move one level up the tree and take the values of Gx and Gy stored at the vertex G. This scheme is particularly efficient if one must calculate the same series $\dot\eta_{xy}$ for several values of the variable u. On the other hand, if we must calculate the values of distinct series $\dot\eta_{xy}$ for the same value of u, then we must store at the vertex G the values of Gu and $\dot Gu$ and use in our calculations the first series in (39).

Of course, the actual calculations are performed for finite subtrees. If we take here a subtree of fixed depth k, then the above lemma gives us an a priori estimate of the summation error. Practice shows, however, that it is more economical to use another tree, which will be defined in the process of the calculations. Namely, if a term of the series in (39) at a vertex G is smaller than the prescribed accuracy ε, then one need not consider the part of the sum corresponding to the subtree with top vertex G because estimates show that the sum taken over that (infinite) subtree has the same order as the term corresponding to G. For this scheme of calculations we can find only a posteriori estimates of the error, that is, we know the accuracy only when the calculations are complete.

3.4. Variational theory. In connection with the solution of equations (15) we are interested in the question of the variations of the Schottky functions (44), (45) and the factors (48) under perturbations of the group $\mathfrak{G}$. The space of (non-normalized) Schottky groups of genus g with fixed system of generators has the natural structure of a $3g$-dimensional complex manifold [21], in which the symmetric groups of § 3.1 form a real submanifold of (real) dimension $2g$. Coordinates in the neighbourhood of a fixed group $\mathfrak{G}_0$ can be defined by the identification of the generators $G_k(u)$ with matrices $\widehat G_k \in \mathrm{SL}_2(\mathbb{R})$, $k = 1, \dots, g$, defined up to a sign.

We consider now Schottky groups $\mathfrak{G}$ close to $\mathfrak{G}_0$ and generated by matrices

$$\widehat G_k = \widehat G_{k0} + \varepsilon \widehat H_k, \qquad k = 1, \dots, g, \tag{54}$$

where $\widehat H_k$ is a matrix tangent to the space SL_2 at the point $\widehat G_{k0}$. For small ε the maps G_k corresponding to (54) generate a Schottky group that has the necessary symmetric form only for a special choice of the perturbations $\widehat H_k$:

$$\widehat H_k = \xi_1^k \begin{Vmatrix} 1 & 2c_{k0} \\ 0 & 1 \end{Vmatrix} + \xi_2^k \begin{Vmatrix} 0 & r_{k0}^2 - c_{k0}^2 \\ 1 & 0 \end{Vmatrix} \quad \text{for } \widehat G_{k0} =: \frac{1}{r_{k0}} \begin{Vmatrix} c_{k0} & c_{k0}^2 - r_{k0}^2 \\ 1 & c_{k0} \end{Vmatrix}, \tag{55}$$

where $\xi_1^k, \xi_2^k \in \mathbb{R}$, $k = 1, \dots, g$. For the perturbed group we can define the following objects:

- $d\eta_{xy}$, a $\mathbf{C}$-normalized Abelian differential of the third kind,
- $d\zeta_1, \dots, d\zeta_g$, $\mathbf{C}$-normalized Abelian differentials of the first kind,
- π_{jk}, $j, k = 1, \dots, g$, the integrals of the Abelian differentials $d\zeta_k$ over the cycles $\mathbf{D}_j := \widetilde{\mathbf{D}}_1 + \dots + \widetilde{\mathbf{D}}_j$ on the perturbed surface.

We shall equip similar objects corresponding to the unperturbed group with subscript 0.

Theorem 7. *Let $\mathfrak{H} = (\widehat{H}_1, \dots, \widehat{H}_g)$ be a tangent vector to the manifold of Schottky groups at a point $\mathfrak{G}_0 = \langle \widehat{G}_{10}, \dots, \widehat{G}_{g0} \rangle$. Then the derivatives in the direction of $\mathfrak{H}$ satisfy the following relations*:

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_p^q (d\eta_{xy} - d\eta^0_{xy}) = \frac{1}{2\pi i} \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \dot\eta^0_{xy}(u) \dot\eta^0_{pq}(u) \operatorname{tr}[\mathcal{M}(u) \widehat{H}_s \widehat{G}_{s0}^{-1}]\, du, \tag{56}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_p^q (d\zeta_k - d\zeta^0_k) = \frac{1}{2\pi i} \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \dot\zeta^0_{k}(u) \dot\eta^0_{pq}(u) \operatorname{tr}[\mathcal{M}(u) \widehat{H}_s \widehat{G}_{s0}^{-1}]\, du, \tag{57}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} (\pi_{jk} - \pi^0_{jk}) = \frac{1}{2\pi i} \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \dot\zeta^0_{j}(u) \dot\zeta^0_{k}(u) \operatorname{tr}[\mathcal{M}(u) \widehat{H}_s \widehat{G}_{s0}^{-1}]\, du, \tag{58}$$

where

$$\mathcal{M}(u) := \begin{Vmatrix} -u & u^2 \\ -1 & u \end{Vmatrix} \in \mathrm{sl}_2; \qquad p, q, x, y \in \mathcal{R}_0.$$

Proof. We verify (56) first. For small ε the fundamental domain $\mathcal{R}_0$ of the group $\mathfrak{G}_0$ lies in the region of discontinuity of the perturbed group $\mathfrak{G}$, therefore the difference of integrals of the third kind $\delta\eta_{xy} := \eta_{xy} - \eta^0_{xy}$ is a (single-valued) holomorphic function in $\mathcal{R}_0$. We have the chain of equalities

$$\begin{aligned} 2\pi i(\delta\eta_{xy}(q) - \delta\eta_{xy}(p)) &= \sum_{s=1}^{g} \int_{\mathbf{C}_{s0} - \mathbf{C}_{-s0}} \delta\eta_{xy}\, d\eta^0_{pq} \\ &= \sum_{s=1}^{g} \int_{\mathbf{C}_{s0} - \mathbf{C}_{-s0}} \eta_{xy}\, d\eta^0_{pq} \end{aligned} \tag{59}$$

$$\left(\text{because } \eta^0_{xy}(G_{s0}u) - \eta^0_{xy}(u) = \text{const and} \int_{\mathbf{C}_{\pm s0}} d\eta^0_{pq} = 0 \right)$$

$$\begin{aligned} &= \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \big(\eta_{xy}(u) - \eta_{xy}(G_{s0}^{-1}u)\big)\, d\eta^0_{pq} \\ &= \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \big(\eta_{xy}(u) - \eta_{xy}(G_s \circ G_{s0}^{-1}u)\big)\, d\eta^0_{pq} \end{aligned} \tag{60}$$

(by the same arguments as above: the difference between $\eta_{xy}(G_s v)$ and $\eta_{xy}(v)$ is a constant)

$$= \varepsilon \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \dot{\eta}_{xy}(u) \operatorname{tr}[\mathcal{M}(u) \widehat{H}_s \widehat{G}_{s0}^{-1}] \, d\eta_{pq}^0(u) + o(\varepsilon) \qquad (61)$$

$$= \varepsilon \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \dot{\eta}_{xy}^0(u) \operatorname{tr}[\mathcal{M}(u) \widehat{H}_s \widehat{G}_{s0}^{-1}] \, d\eta_{pq}^0(u) + o(\varepsilon).$$

In the last equality we use the uniform estimate $\dot{\eta}_{xy} - \dot{\eta}_{xy}^0 = O(\varepsilon)$ on the circle $\mathbf{C}_s$, which can be obtained in the framework of the theory of quasiconformal maps (see Ahlfors [22] and Bers [23] for detail).

Variational formula (57) can be easily obtained using the same scheme as for (56). Nevertheless, we present another derivation of this formula on the basis of a limit transition in (56). We shall look for the limit of the expression

$$\varepsilon^{-1} \int_p^q (d\eta_{xy} - d\eta_{zy}^0) = \varepsilon^{-1} \left[\int_p^q (d\eta_{xy} - d\eta_{xy}^0) + \int_p^q d\eta_{xz}^0 \right] \qquad (62)$$

as $x \to G_k y$, $z \to G_{k0} y$, and $\varepsilon \to 0$ (the limit with respect to ε is taken last). We fix $y \in \mathcal{R}$ and small ε and find the limit as $x \to G_k y$ of the first integral on the right-hand side of (62). We shall use identity (59)–(60) and continue the right-hand side of (60) analytically in x inside the circle $\mathbf{C}_{k0}$: to this end one splits the kth term of the sum into two integrals, and for both intervals one deforms the integration contour $\mathbf{C}_{k0}$ so that it remains all the time in the domain where the Abelian integral $\eta_{xy}(u)$ (respectively, the integral $\eta_{xy}(G_k \circ G_{k0}^{-1} u)$) is single-valued; the kth term on the right-hand side of (60), on passing to the limit as $x \to G_k y$, takes the following form

$$\int_{\mathbf{C}_k'} \zeta_k(u) \, d\eta_{pq}^0(u) - \int_{\mathbf{C}_k''} \zeta_k(G_k \circ G_{k0}^{-1} u) \, d\eta_{pq}^0(u), \qquad (63)$$

where the paths $\mathbf{C}_k'$ and $\mathbf{C}_k''$, which are deformations of $\mathbf{C}_{k0}$, start at the points $G_k(y)$ and $G_{k0}(y)$, respectively. The starting points of integration over $\mathbf{C}_k'$ and $\mathbf{C}_k''$ are important because the value of the Abelian integral ζ_k changes after making a circuit on a contour. We transform the expression (63) as follows:

$$\int_{\mathbf{C}_k} \bigl(\zeta_k(u) - \zeta_k(G_k \circ G_{k0}^{-1} u) \bigr) \, d\eta_{pq}^0(u) + 2\pi i \int_{G_{k0}(y)}^{G_k(y)} d\eta_{pq}^0. \qquad (64)$$

Taking this for the first term on the right-hand side of (62) and cancelling, in view of Riemann's relation for Abelian integrals of the third kind, the last term in (62) with the last term in (64) we arrive at the relation

$$2\pi i \int_p^q (d\zeta_k - d\zeta_k^0) = \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \bigl(\zeta_k(u) - \zeta_k(G_s \circ G_{s0}^{-1} u) \bigr) \, d\eta_{pq}^0(u). \qquad (65)$$

The limit transition with respect to ε is now perfectly similar to (61).

The remaining variational formula (58) can be obtained by a limit transition in (57). For the variation of the period π_{jk} we have

$$\delta\pi_{jk} := \int_{G_j q}^{q} d\zeta_k - \int_{G_{j0}q}^{q} d\zeta_k^0 = \int_{G_{j0}q}^{q} (d\zeta_k - d\zeta_k^0) + \int_{G_j q}^{G_{j0}q} d\zeta_k. \tag{66}$$

In fact, we have already found the first integral on the right-hand side of (66): continuing both parts of (65) analytically in p inside the circle $\mathbf{C}_{j0}$ and passing to the limit as $p \to G_{j0}(q)$ we obtain

$$\begin{aligned} &2\pi i \int_{G_{j0}q}^{q} (d\zeta_k - d\zeta_k^0) \\ &\quad = \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \bigl(\zeta_k(u) - \zeta_k(G_s \circ G_{s0}^{-1} u)\bigr)\, d\zeta_j^0 - 2\pi i \bigl(\zeta_k(G_{j0}q) - \zeta_k(G_j q)\bigr). \end{aligned}$$

Hence

$$2\pi i \delta\pi_{jk} = \sum_{s=1}^{g} \int_{\mathbf{C}_{s0}} \bigl(\zeta_k(u) - \zeta_k(G_s \circ G_{s0}^{-1} u)\bigr)\, d\zeta_j^0(u)$$

and, letting ε approach zero, we obtain the required expression (58) for the derivative.

Remarks. (1) Formula (56) is in fact Hadamard's formula for the variation of the Green's function. Similar variational formulae can be found by Schiffer and Spencer [24], Rauch [25], Ahlfors [22].

(2) The geometric nature of the expressions on the right-hand sides of (56)–(58) is of some interest. In view of the identity $\operatorname{tr}[\mathcal{M}(u)\mathcal{M}(v)] = -(u-v)^2$, these are the values of the periods of the Eichler integral for the quadratic differentials $d\eta_{xy}\, d\eta_{pq}$, $d\zeta_k\, d\eta_{pq}$, and $d\zeta_j\, d\zeta_k$. As regards the connections between variational formulae and Eichler cohomology, see also [26].

3.5. Calculation of variations. The direct calculation of variations on the basis of (56)–(58) is an expensive business because quadrature formulae require that the series be calculated at many points. However, there is a way round, which allows one to obtain the result by summing the series only at $2g-1$ points. We refer here to Hejhal [27], who has explicitly calculated for relative Poincaré Θ_2-series the map

$$\Xi(u)(du)^2 \mapsto \int_{\mathbf{C}_k} \Xi(u)\mathcal{M}(u)\, du, \qquad k = 1, \dots, g, \tag{67}$$

from the space of (meromorphic) quadratic differentials into $\mathrm{sl}_2(\mathbb{C})$, which participates in our variational formulae.

3.5.1. *Quadratic Poincaré series.* With each element $T \in \mathfrak{G}$ distinct from unity with fixed points α (the attracting point) and β (the repelling one) and with dilation coefficient λ, $0 < \lambda < 1$, we can associate a holomorphic quadratic differential

$\Theta_2[R_T](du)^2$ and also a meromorphic quadratic differential $\Theta_2[R_T^{xy}](du)^2$ with poles in the orbits of points $x, y \in \mathcal{D}(\mathfrak{G})$; namely,

$$\begin{aligned} \Theta_2[R_T](du)^2 &:= \sum_{G\in\langle T\rangle\backslash\mathfrak{G}} R_T(Gu)(dG(u))^2, \\ R_T(u) &:= (u-\alpha)^{-2}(u-\beta)^{-2}; \end{aligned} \tag{68}$$

$$\begin{aligned} \Theta_2[R_T^{xy}](du)^2 &:= \sum_{G\in\mathfrak{G}} R_T^{xy}(Gu)(dG(u))^2, \\ R_T^{xy}(u) &:= (u-\alpha)^{-1}(u-\beta)^{-1}(u-x)^{-1}(u-y)^{-1}. \end{aligned} \tag{69}$$

The convergence of (68) is in our case a consequence of the convergence of the series in (41) for the holomorphic differential $d\zeta_k$ and in the general case it follows from classical area estimates [28]; as regards the convergence of the Poincaré Θ-series (69), see, for instance, [28].

Definition. For fixed $k = 1, \dots, g$ we now define the elements $T_j(k)$ of $\mathfrak{G}$ *concomitant* to an element T. If the non-cancellable representation of T in terms of the generators has the form

$$\begin{aligned} T = (G_\bullet \cdots G_k^{\varepsilon_1} \cdots G_k^{\varepsilon_2} \cdots G_k^{\varepsilon_l} \cdots G_*) \cdot (G_\triangle \cdots G_k^{\varepsilon_{l+1}} \cdots G_k^{\varepsilon_s} \cdots G_\triangledown) \\ \cdot (G_*^{-1} \cdots G_k^{-\varepsilon_l} \cdots G_k^{-\varepsilon_2} \cdots G_k^{-\varepsilon_1} \cdots G_\bullet^{-1}), \end{aligned} \tag{70}$$

where we indicate all occurrences of the element G_k, $\varepsilon_j = \pm 1$, and $G_\triangle G_\triangledown \neq 1$, then we set

$$T_j(k) := (G_\bullet \cdots G_k^{\varepsilon_1} \cdots G_k^{\varepsilon_2} \cdots G_k^{\varepsilon_j}) \cdot \begin{cases} G_k^{-1} & \text{for } \varepsilon_j = 1, \\ 1 & \text{for } \varepsilon_j = -1, \end{cases} \quad j = 1, \dots, s. \tag{71}$$

Theorem 8. (1°) *For quadratic Poincaré series the map* (67) *is expressed by a finite sum* (*cf.* [27]):

$$\int_{\mathbf{C}_k} \Theta_2[R_T](u)\mathcal{M}(u)\,du = \frac{2\pi i}{(\alpha-\beta)^3} \sum_{j=l+1}^{s} \varepsilon_j \widehat{T}_j^{-1}(k) \begin{Vmatrix} \alpha+\beta & -2\alpha\beta \\ 2 & -\alpha-\beta \end{Vmatrix} \widehat{T}_j(k); \tag{72}$$

(2°) *if the poles x and y of the quadratic differential $\Theta_2[R_T^{xy}](du)^2$ lie in $\mathcal{R}$, then*

$$\begin{aligned} \int_{\mathbf{C}_k} \Theta_2[R_T^{xy}](u)\mathcal{M}(u)\,du = \frac{2\pi i}{\alpha-\beta}\Bigg[\sum_{j=1}^{l} \varepsilon_j \widehat{T}_j^{-1}\left\{\frac{\mathcal{M}(\alpha)}{(\alpha-x)(\alpha-y)} - \frac{\mathcal{M}(\beta)}{(\beta-x)(\beta-y)}\right\}\widehat{T}_j \\ + \sum_{j=l+1}^{s} \frac{\varepsilon_j}{\lambda^{-1/2}-\lambda^{1/2}}\,\widehat{T}_j^{-1}\left\{\frac{\lambda^{-1/2}\mathcal{M}(\alpha)}{(\alpha-x)(\alpha-y)} - \frac{\lambda^{1/2}\mathcal{M}(\beta)}{(\beta-x)(\beta-y)}\right\}\widehat{T}_j\Bigg]. \end{aligned} \tag{73}$$

Proof. (1°) First of all, we point out Hejhal's key identity

$$\mathcal{M}(Gu) = \widehat{G}\mathcal{M}(u)\widehat{G}^{-1}\dot{G}(u), \qquad G \in \mathfrak{G}, \tag{74}$$

using which we can transform the left-hand side of (72) as follows:

$$\int_{\mathbf{C}_k} \Theta_2[R_T](u)\mathcal{M}(u)\,du = \sum_{G\in\langle T\rangle|\mathfrak{G}} \widehat{G}^{-1}\biggl\{\int_{G\mathbf{C}_k} R_T(u)\mathcal{M}(u)\,du\biggr\}\widehat{G}.$$

The poles of $R_T(u)$ lie inside the circle $G\mathbf{C}_k$ for the following elements G of $\mathfrak{G}$:

TABLE 1

pole	$G\in\mathfrak{G}$	comments
$u=\alpha$	$T_1,\dots,T_l;\ T^mT_{l+1},T^mT_{l+2},\dots,T^mT_s$	$m\geqslant 0$
$u=\beta$	$T_1,\dots,T_l;\ T^mT_s,T^mT_{s-1},\dots,T^mT_{l+1}$	$m<0$

Estimates for the length of the non-cancellable representation in terms of the generators show that the elements $T_1,\dots,T_l;T_{l+1},\dots,T_s$ belong to distinct right cosets by the subgroup $\langle T\rangle$, therefore calculating the residue

$$\begin{aligned}\operatorname*{Res}_{u=\alpha}\bigl(R_T(u)\mathcal{M}(u)\bigr) &= (\alpha-\beta)^{-2}\dot{\mathcal{M}}(\alpha) - 2(\alpha-\beta)^{-3}\mathcal{M}(\alpha)\\ &= (\alpha-\beta)^{-3}\begin{Vmatrix}\alpha+\beta & -2\alpha\beta\\ 2 & -\alpha-\beta\end{Vmatrix},\end{aligned}\tag{75}$$

and the residue of $R_T\mathcal{M}$ at $u=\beta$ we arrive at the right-hand side of (72). Note that the matrix (75) is $\operatorname{Ad}\widehat{T}$-invariant, as the identity

$$\begin{Vmatrix}\alpha+\beta & -2\alpha\beta\\ 2 & -\alpha-\beta\end{Vmatrix} = \binom{\alpha}{1}(1,\,-\beta) + \binom{\beta}{1}(1,\,-\alpha)$$

shows, therefore in the calculations of the right-hand side of (72) we can choose representatives of cosets by the subgroup $\langle T\rangle$ from Table 1.

(2°) We now transform the left-hand side of (73) using Hejhal's identity:

$$\int_{\mathbf{C}_k} \Theta_2[R_T^{xy}](u)\mathcal{M}(u)\,du = \sum_{G\in\mathfrak{G}} \widehat{G}^{-1}\biggl\{\int_{G\mathbf{C}_k} R_T^{xy}(u)\mathcal{M}(u)\,du\biggr\}\widehat{G}.$$

Note that the points $x,y\in\mathcal{R}$ lie outside all the circles $G\mathbf{C}_k$ and the points α and β lie inside the circle $G\mathbf{C}_k$ only for the elements G in Table 1. To take the sum of the quantities $\widehat{G}^{-1}\operatorname{Res}_{u=\alpha,\beta}(R_T^{xy}\mathcal{M})\widehat{G}$ for G in Table 1 we must know the sums of the series

$$\begin{aligned}\sum_{m=0}^{\infty}\widehat{T}^{-m}\mathcal{M}(\alpha)\widehat{T}^m &= \sum_{m=0}^{\infty}\lambda^m\mathcal{M}(\alpha) = \frac{\mathcal{M}(\alpha)}{1-\lambda},\\ \sum_{m=1}^{\infty}\widehat{T}^{m}\mathcal{M}(\beta)\widehat{T}^{-m} &= \sum_{m=1}^{\infty}\lambda^m\mathcal{M}(\beta) = \frac{\lambda\mathcal{M}(\beta)}{1-\lambda}.\end{aligned}$$

Carrying out the calculations we arrive at (73).

3.5.2. *Basis of Poincaré Θ_2-series.* Using Theorem 8 we can calculate Hejhal's map for several (relative) quadratic Poincaré series:

TABLE 2

$T \in \mathfrak{G}$	$\displaystyle\int_{\mathbf{C}_s} \Theta_2[R_T](u)\mathcal{M}(u)\,du$
G_k	$\dfrac{i\pi}{2\alpha^2}\begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix}\delta_{ks}$
$G_k G_s^{-1} G_k$	$-\dfrac{i\pi}{2\alpha^2}\widehat{G}_k\begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix}\widehat{G}_k^{-1}$
$G_s G_k^{-1} G_s$	$\dfrac{i\pi}{2\alpha^2}\left\{\begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix} + \widehat{G}_s\begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix}\widehat{G}_s^{-1}\right\}$

Corollary to Theorem 8. *The quadratic differentials* (68) *with* $T{=}G_1, G_2, \dots, G_g$; $G_1G_2^{-1}G_1, G_1G_3^{-1}G_1, \dots, G_1G_g^{-1}G_1$ *make up a basis in the* $(2g-1)$*-dimensional space of even* (*that is, invariant under the hyperelliptic involution*) *holomorphic quadratic differentials on* M.

Proof. We claim first that all quadratic differentials under consideration are invariant under the involution M, which in our model has the form $u \to -u$. Indeed all the motions T in the statement of the corollary satisfy the relation $T(-u) = -T^{-1}(u)$ (recall that each element $G_1, \dots, G_g$ is a composite of two reflections), therefore

(a) $\alpha(T) + \beta(T) = 0$,
(b) the involution $G(u) \to \widetilde{G}(u) := -G(-u)$ can be extended to the right cosets $\langle T\rangle|\mathfrak{G}$.

These two observations allow us to conclude that $\Theta_2[R_T]$ is an even function of u for the T under consideration.

To prove the completeness of the system of quadratic differentials in question we consider the linear functionals $\langle l_k, \Xi\rangle$ and $\langle m_k, \Xi\rangle$ on holomorphic quadratic differentials $\Xi(u)(du)^2$ that are the entries $(1,2)$ and $(1,1)$, respectively, of the matrix $\displaystyle\int_{\mathbf{C}_k} \Xi(u)\mathcal{M}(u)\,du$. Using Table 2 we make up a matrix of the values of the system of $2g-1$ functionals $\langle l_1, \cdot\,\rangle, \dots, \langle l_g, \cdot\,\rangle$; $\langle m_2, \cdot\,\rangle, \dots, \langle m_g, \cdot\,\rangle$ at the quadratic differentials $\Theta_2[R_T](du)^2$. This matrix is indicated in Fig. 5 below, where we have set $L_k := i\pi\alpha^{-1}(G_k)/2$ and $M_k := i\pi\alpha^{-3}r_1^{-2}c_1(\alpha^2 - \alpha^2(G_1))/2$ for $\alpha := \alpha(G_1G_k^{-1}G_1)$.

If $\alpha^2 = \alpha^2(G_1)$, then $\alpha(G_1)$ must be a fixed point of G_k. Hence the diagonal entries of the matrix are distinct from zero, the matrix is non-degenerate, and the quadratic differentials in question actually make up a basis.

T	$\langle l_1, \cdot \rangle$	$\langle l_2, \cdot \rangle$	$\dots$	$\langle l_g, \cdot \rangle$	$\langle m_2, \cdot \rangle$	$\langle m_3, \cdot \rangle$	$\dots$	$\langle m_g, \cdot \rangle$
G_1	L_1	0	$\dots$	0	0	0	$\dots$	0
G_2	0	L_2	$\dots$	0	0	0	$\dots$	0
$\vdots$	$\vdots$		$\ddots$	$\vdots$	$\vdots$	$\vdots$	0	$\vdots$
G_g	0	0	$\dots$	L_g	0	0	$\dots$	0
$G_1G_2^{-1}G_1$	$*$	$*$	$\dots$	$*$	M_2	0	$\dots$	0
$G_1G_3^{-1}G_1$	$*$	$*$	$\dots$	$*$	0	M_3		0
$\vdots$	$\vdots$			$\vdots$	$\vdots$		$\ddots$	
$G_1G_g^{-1}G_1$	$*$	$*$	$\dots$	$*$	0	0	$\dots$	M_g

,

Figure 5

We shall now show how one can use Poincaré series for the calculation of directional derivatives in Theorem 7. We start with the holomorphic quadratic differentials $d\zeta_j\, d\zeta_k$ used in the calculations of derivatives of the periods π_{jk}. These quadratic differentials are invariant under the involution of M, therefore knowing the coefficients of the expansion of $d\zeta_j\, d\zeta_k$ with respect to the explicit basis of quadratic Poincaré series we can use Table 2 to calculate the map (67) for $\Xi(du)^2 = d\zeta_j\, d\zeta_k$. As regards the calculation of the coefficients of the expansion, it suffices to know a complete collection of effectively calculated functionals on the space of even quadratic differentials.

Lemma 3. *Fix local variables at $2g-1$ points in the hyperelliptic curve M such that no two of them correspond to each other under the involution. Then the values of the quadratic differentials at these points make up a complete collection of functionals on the $(2g-1)$-dimensional space of even holomorphic quadratic differentials.*

Proof. Note that the completeness of the system does not depend on our choice of local variables at the distinguished points $p_1, p_2, \dots, p_{2g-1} \in M$ — a change of coordinates means multiplication of the functionals by constants distinct from zero.

We consider now a second-order element $x \in \mathbb{C}(M)$ [14] that is finite at the distinguished points and an element $w \in \mathbb{C}(M)$ related to x by an equation of the form $w^2 = P_{2g+2}(x)$, where P_{2g+2} is a polynomial of degree $2g+2$ with simple zeros. We can assume without loss of generality that $w \neq 0$ at $p_1, p_2, \dots, p_m$, and we can take x for a local variable at these points; we take w for the local variable at the remaining points $p_{m+1}, p_{m+2}, \dots, p_{2g-1}$. We arrange the values of the corresponding functional at the basis $w^{-2}x^j(dx)^2$ $(j = 0, \dots, 2g-2)$ in the space of even quadratic differentials [14] into a matrix:

	p_k $(k = 1, \dots, m)$	p_k $(k = m+1, \dots, 2g-1)$
$w^{-2}x^j(dx)^2$ $(j = 0, \dots, 2g-2)$	$w^{-2}(p_k)x^j(p_k)$	$4(\dot{P}_{2g+2})^{-2}(p_k)x^j(p_k)$

. (76)

The matrix (76) differs from the Vandermonde matrix $\|x^j(p_k)\|$ by right multiplication by a diagonal matrix. By assumption $x(p_k) \neq x(p_s)$ for $p_k \neq p_s$, therefore the matrix in question is non-degenerate and the functionals make up a basis in the dual space.

In the case of meromorphic quadratic differentials $d\eta_{xy}\,d\eta_{pq}$, $d\zeta_k\,d\eta_{pq}$ we can find a sum of (at most four or at most two, respectively) quadratic differentials of the form (69) with the same singularities and expand the remaining holomorphic part with respect to the explicit basis of relative quadratic Poincaré Θ-series [27].

§ 4. Calculations of Chebyshev polynomials

We shall now show how one can solve in the Schottky model the problems stated in the introduction:

(A) the solution of Abel's equations,
(B) the calculation of T_n and the value of the x-variable.

4.1. Uniformization with slits $\mathbf{A_1}, \mathbf{A_2}, \dots, \mathbf{A_g}$.

4.1.1. *Schottky group.* Let (11), (12) be a point in the moduli space $\mathcal{H}(\mathbb{R}; g, 1)$. We associate with it a symmetric Schottky group $\mathfrak{G}$ as follows. We map the Riemann sphere with slits $[a_1, b_1], [a_2, b_2], \dots, [a_g, b_g], [1, -1]$ conformally onto a circular domain with normalization $\{-1, \infty_\pm, 1\} \to \{0, \pm i, \infty\}$, as in Fig. 6. The slit domain is invariant under complex conjugation, therefore the circular domain is also symmetric relative to the real axis. The last domain generates a Schottky group $\mathfrak{G}$ of the form described in § 3.1.

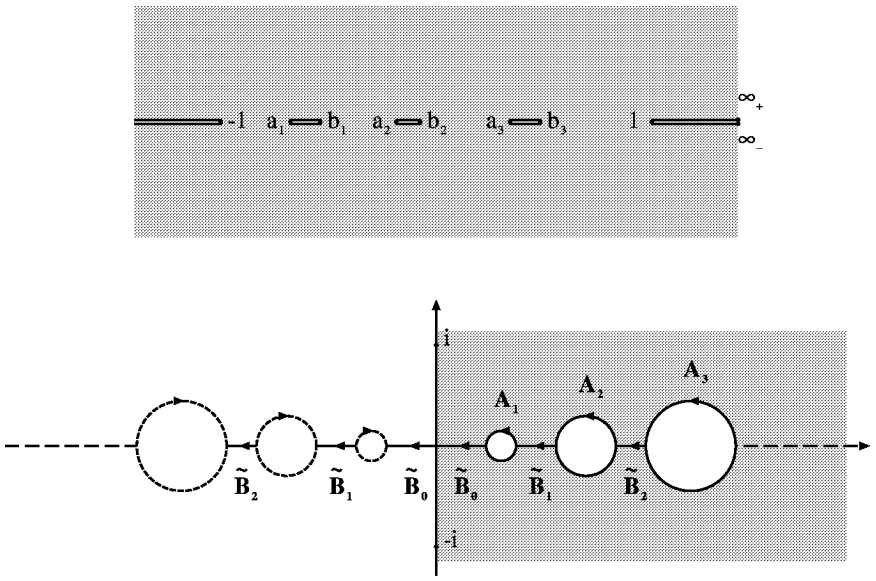


Figure 6. Uniformization of M with slits along the **A**-cycles

The theory of quasiconformal maps [29] — or results on the monodromy map [26] — enable one to assert that the quantities c_k and r_k, the centres and the radii of the discs in (38), satisfying the constraints

$$0 < c_1 - r_1 < c_1 + r_1 < c_2 - r_2 < c_2 + r_2 < \dots < c_g - r_g < c_g + r_g, \qquad (77)$$

make up a global real-analytic system of coordinates in the moduli space $\mathcal{H}(\mathbb{R}; g, 1)$. These coordinates are preferable to the old ones, when the moduli are the ramification points in the surfaces, because the central object of the theory, the **A**-normalized Abelian differential of the third kind $d\eta$ involved in (15) and (16) now has a representation that can be effectively used in calculations:

$$d\eta^{\mathbf{A}} := d\eta_{-ii} = \sum_{G\in\mathfrak{G}} \left\{(Gu+i)^{-1} - (Gu-i)^{-1}\right\} dG(u). \tag{78}$$

4.1.2. *Equations of the cell* $\mathbb{T}(m_0, \dots, m_g)$.

Lemma 4. *Abel's equations* (15) *are equivalent to the following relations*:

$$E_k(i) = \exp\left(i\pi\,\frac{\widetilde{m}_k}{n}\right), \qquad k = 1, \dots, g, \tag{79}$$

where $\widetilde{m}_k := m_0 + m_1 + \cdots + m_{k-1}$.

Proof. (1°) By (15) we obtain

$$2\pi i\,\frac{\widetilde{m}_k}{n} = \int_{\mathbf{B}_k} d\eta = \int_{G_k u}^{u} d\eta_{-ii} \overset{(*)}{=} \int_{-i}^{i} d\eta_{G_k u u} = \int_{-i}^{i} d\zeta_k \overset{(**)}{=} 2\int_{\infty}^{i} d\zeta_k, \tag{80}$$

where the equality $(*)$ follows from Riemann's bilinear relations, the integration of $d\zeta_k$ proceeds along the imaginary axis from $-i$ through ∞ to i; in $(**)$ we use the fact that $d\zeta_k$ is odd with respect to the involution. Dividing both sides of (80) by 2 and exponentiating we obtain (79).

(2°) Conversely,

$$\exp\int_{\mathbf{B}_k} d\eta \overset{(80)}{=} \exp\left(2\int_{\infty}^{i} d\zeta_k\right) \overset{(79)}{=} \exp\left(2\pi i\,\frac{\widetilde{m}_k}{n}\right).$$

The assertion of the lemma now follows from the inequality

$$0 < -i\int_{\mathbf{B}_k} d\eta < 2\pi$$

established in the proof of Lemma 1 for the model (12) on the basis of the conformal map of the upper half-plane onto a 'comb' domain of width 2π. In the Schottky model the quantity $-i\displaystyle\int_{\mathbf{D}_k} d\eta_{-ii}$ is double the angle at which one sees from the point $u = i$ the subset $\mathfrak{G}\mathbf{D}_k = \mathfrak{G}\{\widetilde{\mathbf{D}}_1 \cup \cdots \cup \widetilde{\mathbf{D}}_k\}$ of the real axis (see Fig. 3).

4.1.3. *'Navigation' in the moduli space.* The problem of 'navigation' is to find a descent from an arbitrary point M in the moduli space onto the cell $\mathbb{T}$ described by the system (79). In the final stage of the descent it proves very efficient to use the Newton method, when one linearizes the left-hand sides of equations (79) in the variables $\{c_k, r_k\}_{k=1}^{g}$ of the moduli space using variational Theorem 7. One makes the descent to the closest point in the g-dimensional plane defined by the linear part of (79).

If the initial point M_0 lies far away from the cell, then the 'navigation' becomes a mere 'drift' towards $\mathbb{T}$, for instance, along the trajectories of a suitable dynamical system.

Consider the function of distance from a fixed cell $\mathbb{T}(m_0,\dots,m_g)$ in the coordinate system $\{\mu_k,\nu_k\}_{k=1}^g$ in the moduli space induced by the embedding of $\mathcal{H}$ in the simplex (30) as the submanifold $\{\lambda_k(a,b,c)=0,\ k=1,\dots,g\}$:

$$\Phi(M) := \sum_{k=1}^{g}\left(\mu_k(M) - 2\pi\frac{\widetilde{m}_k}{n}\right)^2. \tag{81}$$

In the coordinates $\{c_k,r_k\}_{k=1}^g$ the same function has the following expression

$$\Phi^{\mathbf{A}}(M) := 4\sum_{k=1}^{g}\left(\operatorname{Arg} E_k(i) - \pi\frac{\widetilde{m}_k}{n}\right)^2 \tag{82}$$

— here we use (80) and the fact that $|E_k(u)|=1$ on the imaginary axis. This distance function, and also its differential, vanishes only for $M\in\mathbb{T}(m_0,\dots,m_g)$. On the complement to $\mathbb{T}(m_0,\dots,m_g)$ we consider the dynamical system

$$\frac{dM}{dt} = -\frac{\nabla\Phi}{|\nabla\Phi|^2}(M), \qquad M\in\mathcal{H}(\mathbb{R};g,1)\setminus\mathbb{T}, \tag{83}$$

which, unfortunately, is not invariant under changes of variables. The derivative of Φ along a trajectory of (83) is -1, so that starting from M_0 we can arrive at the submanifold in time $\Phi(M_0)$. However, we cannot be sure that the trajectory does not leave the moduli space in even shorter time. To understand whether *all trajectories of the dynamical system* (83) *written with respect to the variables* $\{c_k,r_k\}_{k=1}^g$ *intersect the cell* $\mathbb{T}(m_0,\dots,m_g)$ one must analyze in detail the behaviour of the diffeomorphism $\{c_k,r_k\}_{k=1}^g \leftrightarrow \{\mu_k,\nu_k\}_{k=1}^g$ near the boundary of the moduli space. Thus far, this remains an open question.

Finding ourselves on a $\mathbb{T}$-manifold we can move along it because we can effectively calculate the tangent plane at each point; in this manner we can attain each point in the manifold.

4.1.4. *Parametric representation of Chebyshev polynomials.* It remains to recover the Chebyshev polynomials corresponding to points in the manifold of support sets. We start with the x-variable, which we regard as a function on the surface $M\in\mathcal{H}(\mathbb{R};g,1)$. We consider a second-order element $x_1\in\mathbb{C}(M)$ taking the values $0,1,\infty$ for $x=-1,\infty,1$, respectively, so that

$$x = \frac{x_1+1}{x_1-1}. \tag{84}$$

It is easy to see that $\dfrac{dx_1}{x_1}(u) = 2\,d\eta_{0\infty}(u)$, therefore

$$x_1(u) = \exp\left(2\int_i^u d\eta_{0\infty}\right) = (u,i;0,\infty)^2$$

and $x(u)$ can be recovered by formulae (84) and (44). The expression for T_n is an immediate consequence of (16):

$$T_n = \frac{1}{2}\big[(u,\infty;-i,i)^n + (u,\infty;i,-i)^n\big].$$

4.2. Uniformization with slits $\widetilde{\mathbf{B}}_1,\dots,\widetilde{\mathbf{B}}_g$. Carrying out the Schottky uniformization with slits along the $\widetilde{\mathbf{B}}$-cycles we obtain more cumbersome formulae.

4.2.1. *Schottky group.* We fix a surface M of the form (11), (12) and map the extended plane with slits $[-1,a_1],[b_1,a_2],[b_2,a_3],\dots,[b_g,1]$ conformally, with normalization $\{-1,a_1,\infty\}\to\{\infty,0,1\}$, onto a circular domain (see Fig. 7).

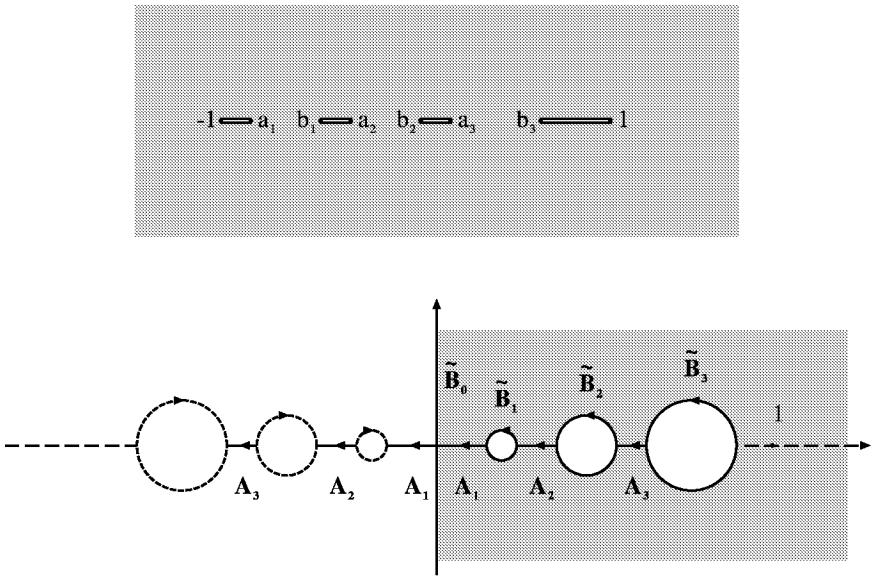


Figure 7. Uniformization of M with slits along the $\widetilde{\mathbf{B}}$-cycles

As before, this circular domain gives rise to a symmetric Schottky group, which gives us another system of global real-analytic coordinates $\{c_k,r_k\}_{k=1}^g$ in the moduli space,

$$0<c_1-r_1<c_1+r_1<c_2-r_2<c_2+r_2<\dots<c_g-r_g<c_g+r_g<1; \qquad (85)$$

they are, of course, different from the variables in (77).

4.2.2. *Equations of the cell* $\mathbb{T}(m_0,\dots,m_g)$.

Lemma 5. *Abel's equations* (15) *are equivalent to the following relations*:

$$E_k^{2n}(1) = E_{1k}^{m_1}E_{2k}^{m_2}\cdots E_{gk}^{m_g}, \qquad k=1,\dots,g. \qquad (86)$$

Proof. The $\mathbf{A}$-normalized differential of the third kind $d\eta$ has the following form in the model under consideration:

$$d\eta^{\mathbf{B}} := d\eta_{-11} + \sum_{j=1}^{g} \gamma_j \, d\zeta_j, \tag{87}$$

where the coefficients γ_j are the solutions of the linear system

$$2\int_{\infty}^{1} d\zeta_s + \sum_{j=1}^{g} \gamma_j \int_{\widetilde{\mathbf{A}}_s} d\zeta_j = 0, \qquad s = 1, \dots, g. \tag{88}$$

Indeed, $\mathbf{A}$-normalization is the same as $\widetilde{\mathbf{A}}$-normalization and

$$\int_{\widetilde{\mathbf{A}}_s} d\eta_{-11} = \int_{G_s u}^{u} d\eta_{-11} = \int_{-1}^{1} d\eta_{G_s u u} = \int_{-1}^{1} d\zeta_s = 2\int_{\infty}^{1} d\zeta_s.$$

The differential (87) satisfying Abel's equations has the coefficients $\gamma_j = m_j/n$, which brings us to the form (17) of these equations. Equations (88) are real, therefore, we arrive at equivalent equations (86) by exponentiating.

4.2.3. *'Navigation' in the moduli space.* The above-introduced function Φ measuring the distance to the cell $\mathbb{T}$ embedded in the space $\mathcal{H}$ has in the new variables (85) the following form:

$$\Phi^{\mathbf{B}}(M) = (2\pi)^2 \sum_{k=1}^{g} \biggl(\sum_{s=k}^{g} \Bigl(\gamma_s - \frac{m_s}{n} \Bigr) \biggr)^2, \tag{89}$$

where the $\gamma_s(M)$ are the solutions of the linear system (88). Descending along gradient trajectories of $\Phi^{\mathbf{B}}$ we arrive at the manifold of support sets $\mathbb{T}$ (again, this is an experimentally established fact) and can move further along this manifold.

4.2.4. *Recovering the polynomial.* A parametric representation of the Chebyshev polynomial $T_n(\,\cdot\,, x)$ is as follows:

$$\begin{gathered} T_n = \frac{1}{2}(V + V^{-1}), \\ V(u) := (-1)^n (u, \infty; -1, 1)^n E_1^{m_1}(u) E_2^{m_2}(u) \cdots E_g^{m_g}(u), \end{gathered} \tag{90}$$

$$x = \frac{x_1 - a_1}{1 - x_1}, \quad x_1(u) = (u, 1; 0, \infty)^2, \quad a_1 = 2x_1(c_g + r_g) - 1. \tag{91}$$

Remark. The normalization that we used in § 4.2 is not completely symmetric relative to the components of the support set: the cycle $\widetilde{\mathbf{B}}_0$ is distinguished. We could use the normalization $\{b_s, a_{s+1}, \infty\} \to \{\infty, 0, 1\}$ instead, which would slightly change formulae (87), (86), (90), and (91).

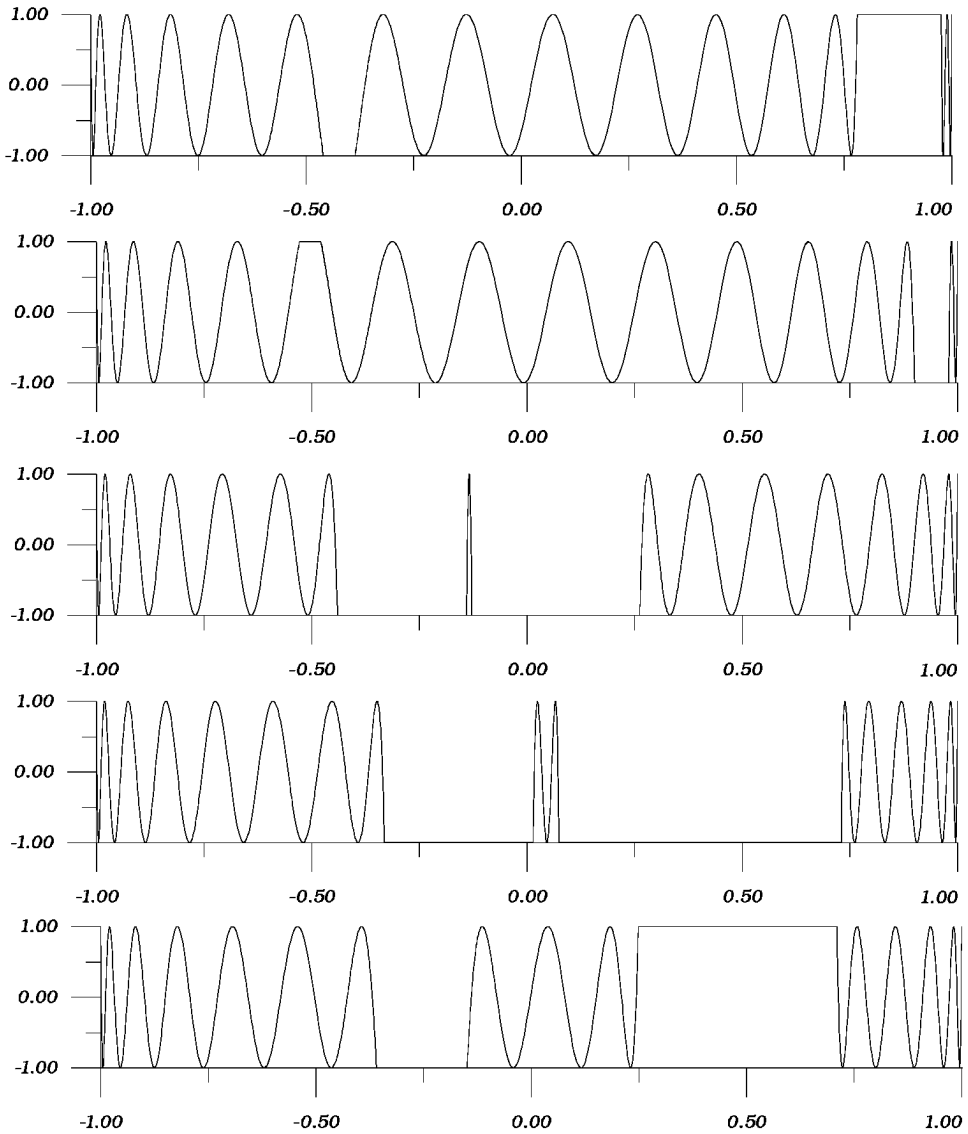


Figure 8. The Chebyshev polynomials for $g = 2$, $n = 30$, and the partitionings $11 + 15 + 4$; $10 + 17 + 3$; $13 + 2 + 15$; $15 + 4 + 11$; and $13 + 7 + 10$

4.3. Graphs of Chebyshev polynomials. Using the formulae presented in this paper this author has written a computer program SCHOTTKY calculating Chebyshev polynomials least deviating from zero on systems of intervals. The input data are the integers g, $m_0, \ldots, m_g$ specifying the cell $\mathbb{T}$; the initial point $\{c_k^0, r_k^0\}_{k=1}^g$ in the moduli space (85); and accuracy ε. The program performs with accuracy ε the gradient descent onto the manifold $\mathbb{T}(m_0, \ldots, m_g)$ in the moduli space and finds a parametric representation for the corresponding Chebyshev polynomials by formulae (90) and (91). We present the graphs of several polynomials in Fig. 8.

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